v1.2

Learning Curve Analysis

How to account for cost improvement

> "O this learning, what a thing it is!"
> -Gremio, *The Taming of the Shrew*, Act I, Scene iv
>
> "If thou wert my fool, nuncle, I'd have thee beaten for being old before thy time. … Thou shouldst not have been old till thou hadst been wise."
> -The Fool, *King Lear*, Act I, Scene v

Learning Curve Overview

- Key Ideas
  - Cost Improvement
    - Touch Labor
    - Constant percent decrease for each doubling in quantity
  - Unit and CUMAV costs
  - Unit and Lot data
    - Lot Midpoint (LMP)

$$Y = aX^{b}$$

- Practical Applications
  - Historical Learning Curve Determination
  - Future Learning Curve Projection
  - Production Break Effects
  - New Work Effects
  - Production Rate Effects

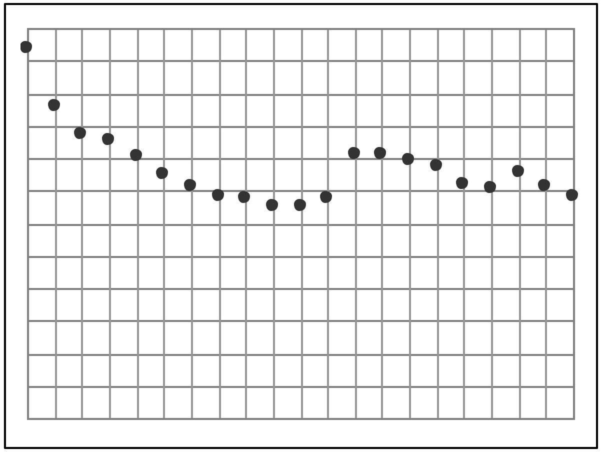
- Analytical Constructs
  - Power Functions
  - Logarithms
    - Log-log transformation
  - Cumulative averages
  - Approximating discrete sums with definite integrals!

- Related Topics
  - Data Normalization and Analysis 4 6
  - 8 Regression Analysis
  - Manufacturing Cost Estimating 11

Learning Curve Within The Cost Estimating Framework

Past

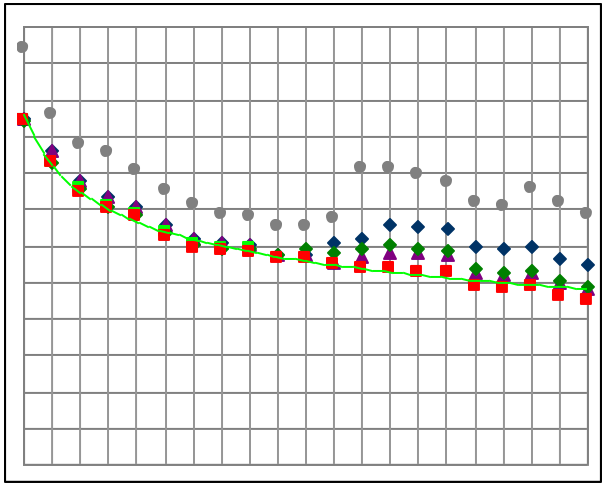
Understanding your historical data



Historical data for multiple units or lots

Present

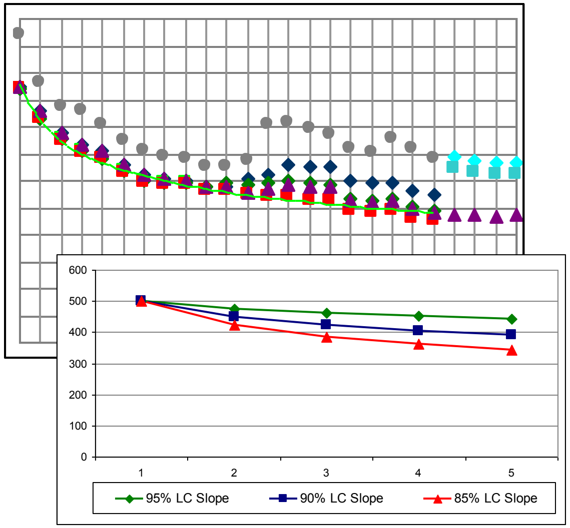
Developing estimating tools



Learning Curve (best fit)

Future

Estimating the new system



Projecting costs for future units or lots

Learning Curve Outline

- Core Knowledge
  - Learning Curve Theory
    - What is a Learning Curve?
    - Learning Curve Concepts
    - Cumulative Average Learning Curve Theory (CUMAV)
    - Unit Learning Curve Theory (ULC)
    - T1 and Lot Midpoint
  - Learning Curve Application
    - Choosing a Learning Curve Theory
    - Learning Curves for New Programs
    - Industry Average Curves
    - Factors Affecting Slope
- Summary
- Sample test Questions
- Related and Advanced Topics

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What is a Learning Curve?

- **Learning Curve**: Constant rate of reduction in touch labor costs for each doubling in quantity
  - Assumes no major change in product design, production processes, workforce composition, and interval between units
  - Since the original theories, learning curves have been applied to material, total cost, total labor hours, etc.

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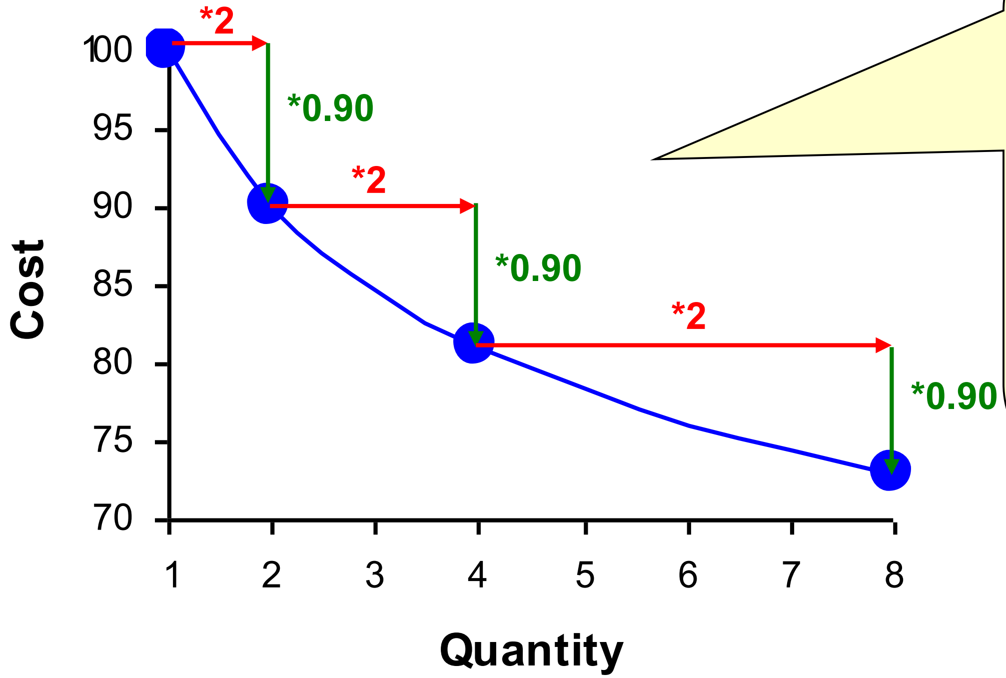
AKA Cost Improvement Curve (CIC), Cost/Quantity Relationship, Manufacturing Progress Function, Experience Curve, Product Improvement Function

- Layman's term "learning curve"
  - "Steep" = rapid improvement, "getting up to speed"

"Factors Affecting the Cost of Airplanes," T.P. Wright, Journal of the Aeronautical Sciences, Feb, 1936.

Learning Curve Concepts

- Learning Curve theory can be shown graphically as follows:



Each time quantity doubles, cost decreases by a constant percentage. "Learning Curve Slope" or "LCS" (e.g., 90%) is defined to be "100% - Percent of Cost Reduction."

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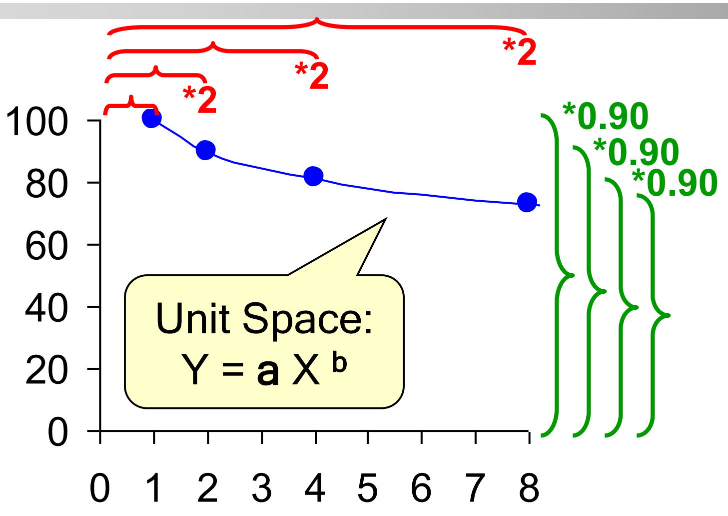
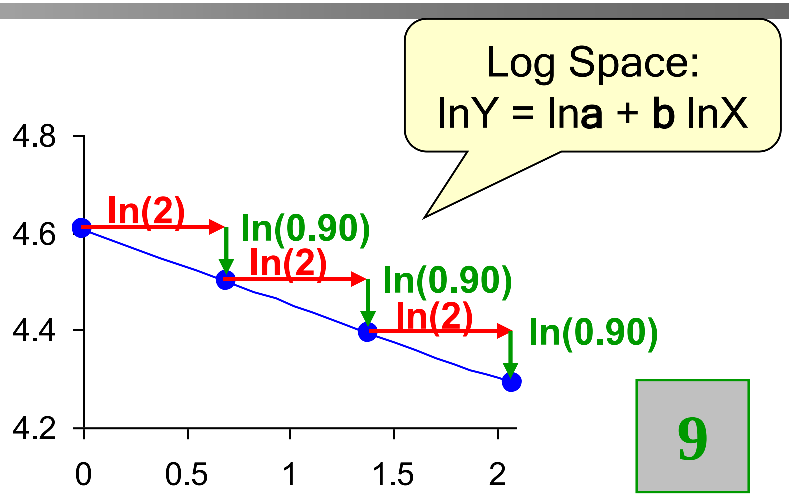
- The implied curve is a function of the form $Y = a X^b$
- Y can be unit or cumulative average cost

Learning Curve Concepts - Slope



Unit Space:
Y = **a** X ^{b}



Log Space:
lnY = ln**a** + **b** lnX

9

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- When transformed to log-space, the learning curve becomes a line
- There are two slopes referred to in learning curve analysis
  - b, which is the slope of the log-space line:

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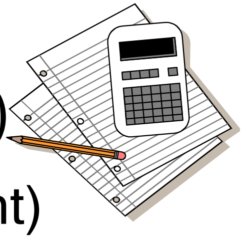
$$b = \ln(LCS)/\ln(2) = \log_2 LCS \quad = LOG(LCS,2)$$



  - LCS, which is the difference between 100% (no improvement) and percentage decrease in cost (e.g., 10% above) every time x doubles

$$LCS = e^{b\ln(2)} = 2^b$$

Learning Curve Concepts - Equation

- Below is the mathematical verification of the learning curve effect

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- Start with the basic LC equation

$$\boxed{Y = aX^{b}}$$

Tip: These are the two main formulae to memorize

double the quantity

$$a(2X)^{b} = a(2^{b})(X^{b}) = 2^{b}(aX^{b}) = 2^{b}Y$$

and since LCS = 2^{b} ,

$$2^{b}Y = LCS \cdot Y$$

When the quantity (X) doubles, the cost (Y) is multiplied by the LCS, as expected

Learning Curve Concepts – Terminology

- Learning Curve Data
  - **Unit *Data*** – Individual unit costs (UC_i)
  - **CUMAV *Data*** – Cumulative average unit costs ($CAUC_k$)
  - **Lot *Data*** – Total cost ($Lot_{F,L}$) and First and Last unit numbers of each lot
- Learning Curve Theory
  - **Unit *Theory*** (ULC) – Unit Cost follows a learning curve
  - **CUMAV *Theory*** – CAUC follows a learning curve
- Axes (for scatterplots)
  - **Unit *Space*** – X = Unit #, Y = Cost (or Labor Hours)
  - **Log *Space*** – X = ln(Unit #), Y = ln(Cost) (or ln(Hrs))

Background - Normalizations

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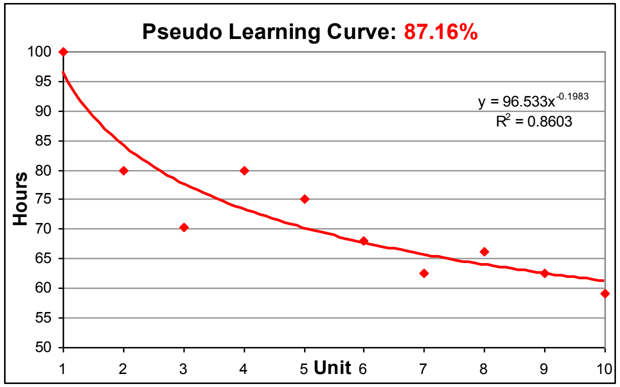
- It is imperative that the data be normalized for all possible effects before any Learning Curve conclusions are drawn
  - Labor Force Composition
  - Change Orders (COs) or Engineering Change Proposals (ECPs)
  - Percent Overlap between successive units



- With these effects present, learning effects often still appear
  - Without normalization, these effects can cause errors in learning slopes in the range of 6 to 18 percentage points
  - A curve derived from such data will hereafter be referred to as a "<u>pseudo learning curve</u>"
    - A pseudo learning curve will often demonstrate statistical significance
    - This does not make this curve valid for estimating
- Pseudo learning curves are most easily discovered graphically

Competing LC Theories

- CUMAV theory came first, but has come into question with many estimators
  - CUMAV has a smoothing effect by making the dependent variable a calculated variable (CAUC)
    - Produces artificially favorable statistics
    - Obscures the variation of the *observed* variable
    - Makes it harder to quantify the error of what we're trying to predict – unit (or lot) cost!
- CUMAV is easier to calculate for Lot data
  - *Not* a compelling reason to use – computers make computations easy!
- We cover CUMAV first for historical and pedagogical reasons
  - A more satisfactory alternative version of CUMAV is presented later

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"CUMAV vs. Unit: Is Cumulative Average vs. Unit Learning Curve Theory a Fair Fight?" B. L. Cullis, R. L. Coleman, P. J. Braxton, J. T. McQueston, SCEA/ISPA 2008.

Cumulative Average Theory Overview

- First asserted by T.P. Wright in 1936
  - Manufacturing data from a small two seat aircraft
- Cumulative Average (CUMAV) Theory predicts learning effects by computing the **cumulative average unit cost** across X units

$$Y = aX^{b}$$

Y = **Cumulative Average** Cost of X Units ($CAUC_X$)

a = Theoretical First Unit Cost (T1)

X = **Cumulative Number** of Units Produced

b = $\log_2$(LCS), a constant reflecting the rate of cost decrease from unit to unit

AKA Wright Curves

T.P. Wright, 1936.

12 13 17

Downside of CUMAV-Direct Method

- CUMAV Theory by Direct Method: $CAUC = T_1Q^b$
- Given the LTCs of k consecutive lots, the CAUC is obtained by
  - $CAUC_1 = LTC_1/LQ_1$,
  - $CAUC_2 = (LTC_1+LTC_2)/(LQ_1+LQ_2)$, …,
  - $CAUC_k = (LTC_1+LTC_2+…+ LTC_k)/(LQ_1+LQ_2+…+LQ_k)$

  LTC_i is the lot total cost (LTC) of lot i; LQ_i is the lot quantity of lot i; $CAUC_i$ is the cumulative average unit cost through lot i

LTC = Lot Total Cost; LQ = Lot Quantity

"Accuracy Matters: Selecting a Lot-Based CIC," S. Hu and A. Smith, SCEA/ISPA 2012.

- "**Data smoothing**" is problematic
  - The dependent variable is truly "dependent" because every observation depends upon all previous observations except for lot 1, which violates the assumptions of OLS regression analysis
  - It generates artificially tight **goodness-of-fit** measures
  - **Outliers** cannot be easily identified for further scrutiny

It is impossible to calculate CAUC if there is a missing or concurrent lot

NEW!

CUMAV Iterative Method

- LTC = $T_1((PQ+LQ)^{b+1} - (PQ)^{b+1})$; AUC = $T_1((PQ+LQ)^{b+1} - (PQ)^{b+1})/LQ$
  - Requires nonlinear regression to derive a solution
  - This equation form (along with other drivers) was used to analyze the CIC slopes for the Unmanned Space Vehicle Cost Model, 8th Edition (USCM8)

PQ = Prior Quantity; LQ = Lot Quantity

- AUC = $T_1(LPP)^b$
  - The lot plot point (LPP) is given by
  - This equation is a log-linear model and the solution can be obtained by OLS in **log** space using an **iterative** approach; the solution steps are the same as the steps for deriving the ULC, with LMP replaced by LPP

$$LPP = \left(\frac{(PQ+LQ)^{b+1} - (PQ)^{b+1}}{LQ} \right)^{1/b}$$

- **No data smoothing**: every data point is treated independently
- The **goodness-of-fit** measures generated by the iterative method are more reliable and realistic than those generated by the direct method
- **Outliers** are easily identified for further scrutiny

"Accuracy Matters: Selecting a Lot-Based CIC," S. Hu and A. Smith, SCEA/ISPA 2012.

CUMAV-Iterative Method can handle missing, nonconsecutive, and concurrent lots

NEW!

Unit Learning Curve Theory Overview

- Unit Learning Curve (ULC) Theory determines learning effects by calculating the individual costs of each production unit, and then summing them together for a total production cost

$$Y = aX^b$$

Note: Same equation as CUMAV, but different definition for Y

Y = **Cost of the Xth unit**

a = Theoretical First Unit Cost (T1)

X = **Sequential unit number** of unit being calculated

b = $\log_2$(LCS), a constant reflecting the rate of cost decrease from unit to unit

7

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AKA Crawford Curves, Unit Theory (UT)

J.R. Crawford, Lockheed.

Theoretical First Unit

- The <u>theoretical</u> first unit cost (T1) is a value determined by analysis in order to best fit the available historical data
- It is not identical to the <u>actual</u> first unit cost, which may be different than predicted, as is the case for every unit
- A T1 adder is said to occur when the actual first unit cost is <u>significantly</u> greater than predicted
  - When first unit produced is actually part prototype and part production unit
  - Common in shipbuilding, may equate to a prototype cost in other commodities

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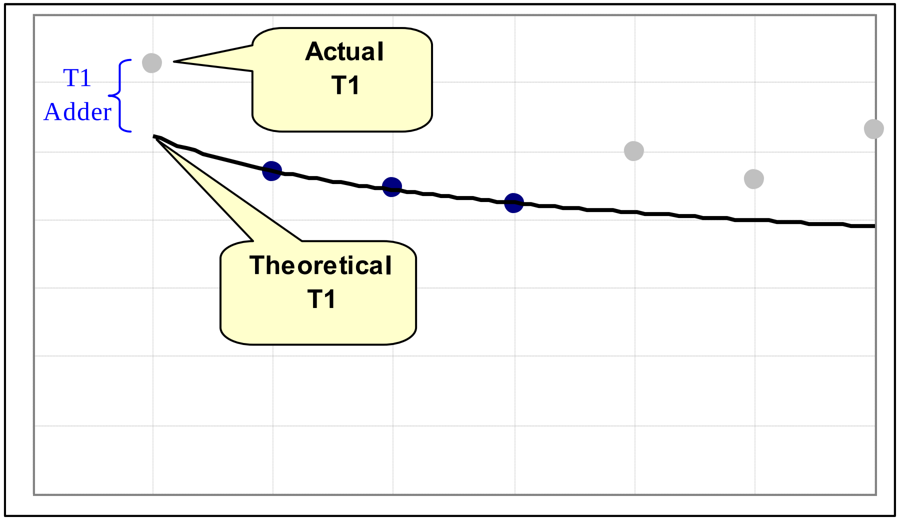
Warning: If a T1 adder is expected, then a near-complete third unit is the bare minimum for accurate learning curve fitting



Lot Midpoint Definition

- Before we begin ULC with lot data, we must clearly define the Lot Midpoint (LMP)
  - Lot midpoint is *only* used for ULC, not CUMAV
- The lot midpoint is the unit at which the average cost of the lot occurs
  - In plain English, when you plug the lot midpoint into the learning curve formula, you get the lot average unit cost
  - It is *not* the same as the average unit number of the lot
  - It is *not* necessarily an integer

Tip: Lot Midpoint (LMP) gives the x-coordinate of the "best" or "most representative" point for the entire lot

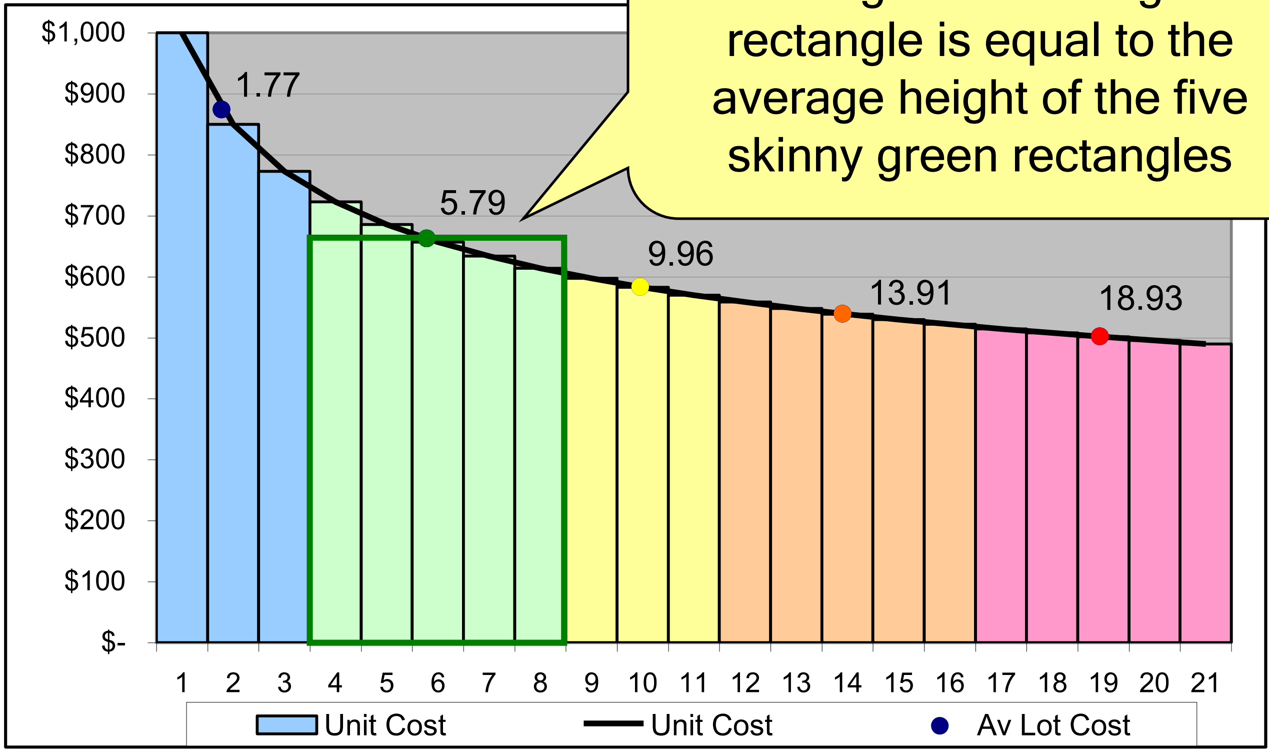
$$AUC = A(LMP)^b = \frac{\sum_{i=F}^{L} Ai^b}{N} \Rightarrow LMP = \left(\frac{\sum_{i=F}^{L} i^b}{N}\right)^{1/b}$$

Note that the computation of LMP is dependent upon b

v1.2

Lot Midpoint Illustration

Unit Cost	Av Lot Cost	LMP
$ 1,000.00	$ 874.30	1.77
$ 850.00	$ 874.30	1.77
$ 772.91	$ 874.30	1.77
$ 722.50	$ 662.59	5.79
$ 685.67	$ 662.59	5.79
$ 656.98	$ 662.59	5.79
$ 633.66	$ 662.59	5.79
$ 614.13	$ 662.59	5.79
$ 597.40	$ 583.39	9.96
$ 582.82	$ 583.39	9.96
$ 569.94	$ 583.39	9.96
$ 558.43	$ 539.41	13.91
$ 548.05	$ 539.41	13.91
$ 538.61	$ 539.41	13.91
$ 529.97	$ 539.41	13.91
$ 522.01	$ 539.41	13.91
$ 514.64	$ 501.80	18.93
$ 507.79	$ 501.80	18.93
$ 501.39	$ 501.80	18.93
$ 495.40	$ 501.80	18.93
$ 489.76	$ 501.80	18.93



The height of the fat green rectangle is equal to the average height of the five skinny green rectangles

Tip: Lot Midpoint (LMP) is the x-coordinate where the rectangle whose area (over the same base) equals the sum of the unit costs (bars) for that lot intersects the learning curve

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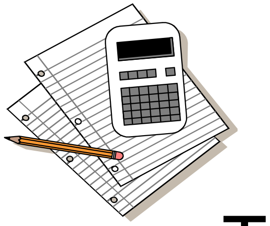
Lot Midpoint Heuristic

- “Best” Lot Midpoint Heuristic:



$$LMP \approx \frac{\sqrt{FL} + {}^{(F+L)}\!/_{2}}{2} = \frac{F+L+2\sqrt{FL}}{4}$$

$$= AVERAGE(GEOMEAN(F,L), AVERAGE(F,L))$$

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- This heuristic provides an *estimate* of the LMP
  - It does not provide the “true” LMP
  - Relatively easy to calculate
  - This in turn provides an estimate for **b**, which is a required parameter in the more accurate LMP equation, which will be presented shortly

“Evaluation of an Alternative Estimator of Learning Curve Lot Midpoints,” Daniel Nussbaum, Journal of Cost Analysis, SCEA, Spring 1994.

★★★★★

Lot Midpoint Approximation

Unit Cost	Av Lot Cost	LMP
$ 1,000.00	$ 874.30	1.77
$ 850.00	$ 874.30	1.77
$ 772.91	$ 874.30	1.77
$ 722.50	$ 662.59	5.79
$ 685.67	$ 662.59	5.79
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$ 529.97	$ 539.41	13.91
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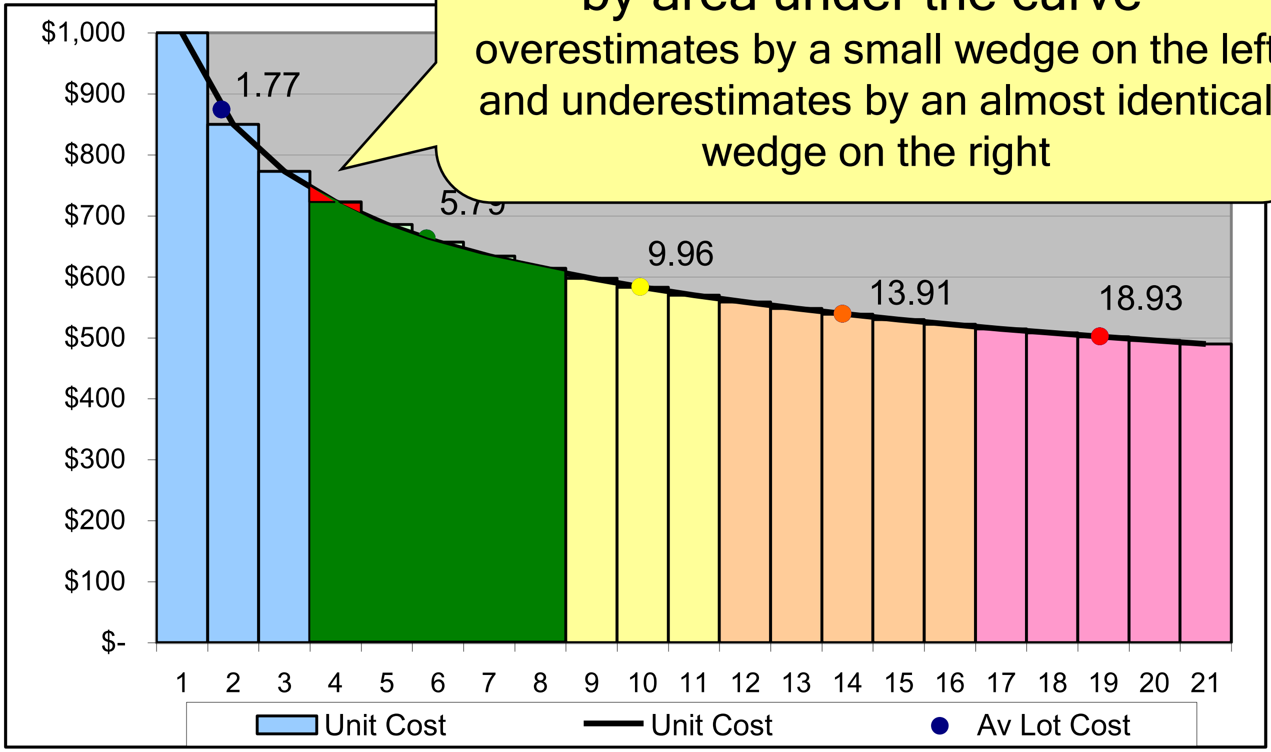
$ 507.79	$ 501.80	18.93
$ 501.39	$ 501.80	18.93
$ 495.40	$ 501.80	18.93
$ 489.76	$ 501.80	18.93



Cost of each lot is approximated by area under the curve – overestimates by a small wedge on the left and underestimates by an almost identical wedge on the right

7

$$LMP \approx \left[\frac{\left(L+\frac{1}{2}\right)^{b+1} - \left(F-\frac{1}{2}\right)^{b+1}}{N(b+1)} \right]^{(1/b)}$$

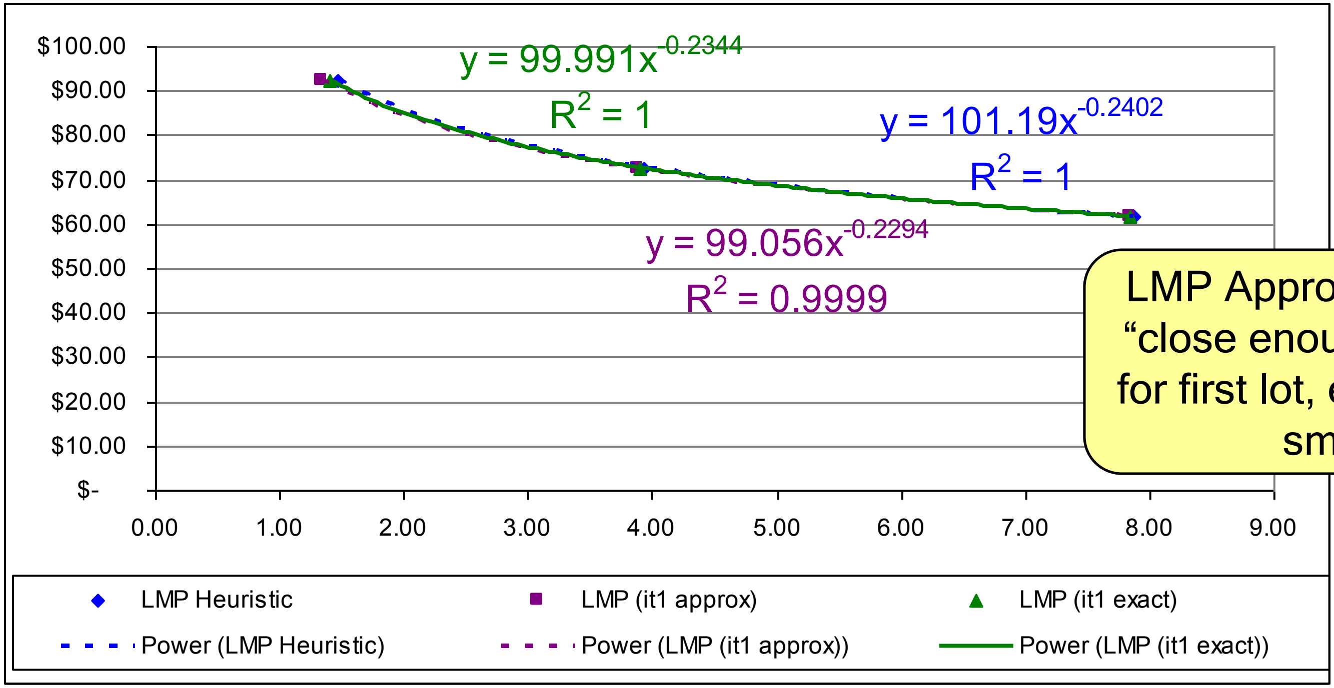
AKA Asher's Approximation

"The Manufacturing Progress Function," R. W. Conway and A. Schultz, Jr., Journal of Industrial Engineering, Jan-Feb 1959, pp. 39-54.

"Accuracy Matters: Selecting a Lot-Based CIC," S. Hu and A. Smith, SCEA/ISPA June 2012.

Lot Midpoint Approximation Effect

Lot	F	L	N	Cost ($)	AUC ($)	LMP Heuristic	LMP (it1 approx)	LMP (it1 exact)
1	1	2	2	$ 185.00	$ 92.50	1.46	1.34	1.39
2	3	5	3	$ 218.11	$ 72.70	3.94	3.88	3.89
3	6	10	5	$ 308.50	$ 61.70	7.87	7.83	7.84



LMP Approximation is "close enough" except for first lot, especially if small

Lot Midpoint Rules of Thumb

- There are some rules of thumb to make estimating a lot midpoint easier

N = number of units in lot

- Rule of Thumb #1
  - When the first lot has fewer than 10 units, the midpoint is about N/2
  - When the first lot has 10 or more units, the midpoint is about N/3
  - Successive lot midpoints are about N/2 + total previous units produced
- Rule of Thumb #2
  - When the first lot has 10 or fewer units, the midpoint is about 0.5*N
  - When the first lot has more than 10 units, the midpoint is about 0.3*N + 1
  - Successive lot midpoints are about 0.5*N + total previous units produced
- Rule of Thumb #3
  - When the first lot has fewer than 10 units, the midpoint is about 0.5*N
  - When the first lot has 10 or more units, the midpoint is about 0.3*N
  - Successive lot midpoints are about 0.5*N + total previous units produced

Tip: Rules of thumb are good for “cocktail napkin” calculations … when you have a computer, use more accurate methods

Choosing a Learning Curve Theory

- Choose the model that best fits the data you have available

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  - Consider regression statistics (R^2, t and F)
  - Although some industries, commodities, or situations may prefer one theory over another, it is up to the analyst to make the proper decision regarding the best learning curve

Warning: Statistics of the regression inherently favor CUMAV over Unit theory!

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"CUMAV vs. Unit: Is Cumulative Average vs. Unit Learning Curve Theory a Fair Fight?" B. L. Cullis, R. L. Coleman, P. J. Braxton, J. T. McQueston, SCEA/ISPA 2008.

★★★★★

Industry Average Curves

Military Examples

- OH-6A Observation Helicopter - 86%
- M113 APC - 93.3%
- Shillelagh Missile - 70.5%
- Airborne Forward Looking Radar - 95.4%
- 7.62mm "Minigun" - 94.7%
- Titan III C Launch Vehicle - 85.4%

Civilian Examples

- Model-T Ford - 86%
- Aircraft Assembly - 80%
- Steel Production - 79%
- Hand-held Calculators - 74%
- MOS Dynamic RAM - 68%
- Electric power generation - 95%

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Warning: These LCs omit the learning model that was used to derive them, and so can only be taken as general indicators

Learning Curve Rule of Thumb

- The steepest learning occurs when the production process is touch labor intensive with little automation
  - Corresponds with the lowest LCS and b
- The flattest learning occurs when the production process is highly automated
  - Corresponds with the highest LCS and b

These are generally believed to be true but may conflict with real-world examples of learning

Application of Learning Curves in Cost Estimating

- Thus far we have discussed (1) determining T1 and LCS from actual data for ongoing programs

2

- By contrast, we often need to determine what LCS to use for a new program
  - Slope estimates are usually determined by analogy to a similar system, modified based on sound analysis
- (2a) New program: Estimate T1, analogy LCS
- (2b) Second production line: Borrow T1, borrow or analogy LCS
- (2c) Production line restart: Same T1, same LCS

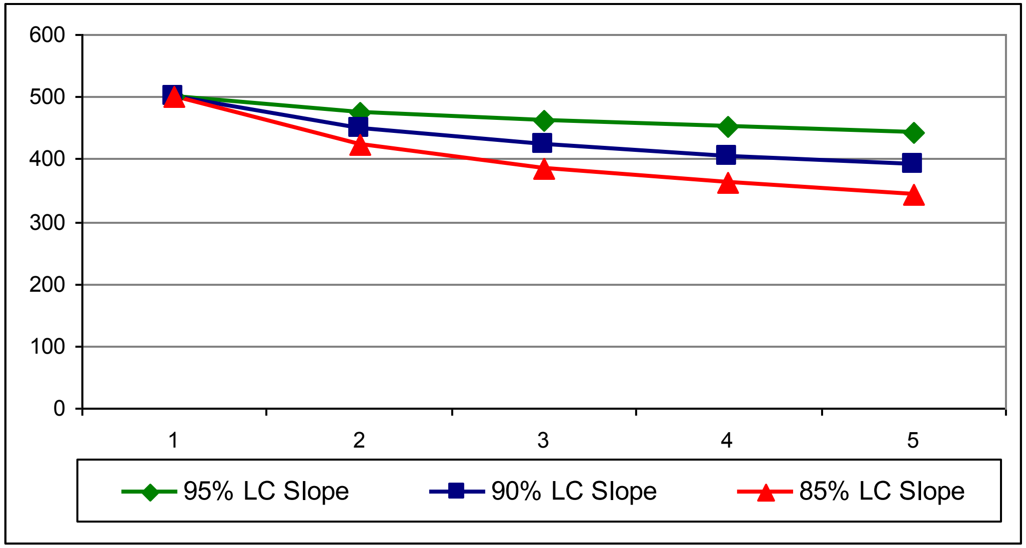
7

Tip: The projected LCS is often the parameter which has the single biggest effect on the production portion of a life cycle cost estimate. Care must be taken to justify thoroughly the LCS value used.

Choice of Learning Curve Slope and Its Effect on an Estimate

7

- The choice of a Learning Curve Slope can greatly affect the value of estimates (especially on follow units)
  - In this example a 10-percentage-point change in LCS causes a 14% difference in Total Cost
  - Also causes a 23% difference in the cost of the 5th Unit



		Unit					Unit 5 delta		Total delta	
	T1 Value ($M)	2	3	4	5	Total	$M	%	$M	%
95% LC Slope	500.0	475.0	461.0	451.3	443.9	2331.1	0.0	0.0%	0.0	0.0%
90% LC Slope	500.0	450.0	423.1	405.0	391.5	2169.6	52.4	11.8%	161.5	6.9%
85% LC Slope	500.0	425.0	386.5	361.3	342.8	2015.5	101.0	22.8%	315.5	13.5%

Learning Curve Summary

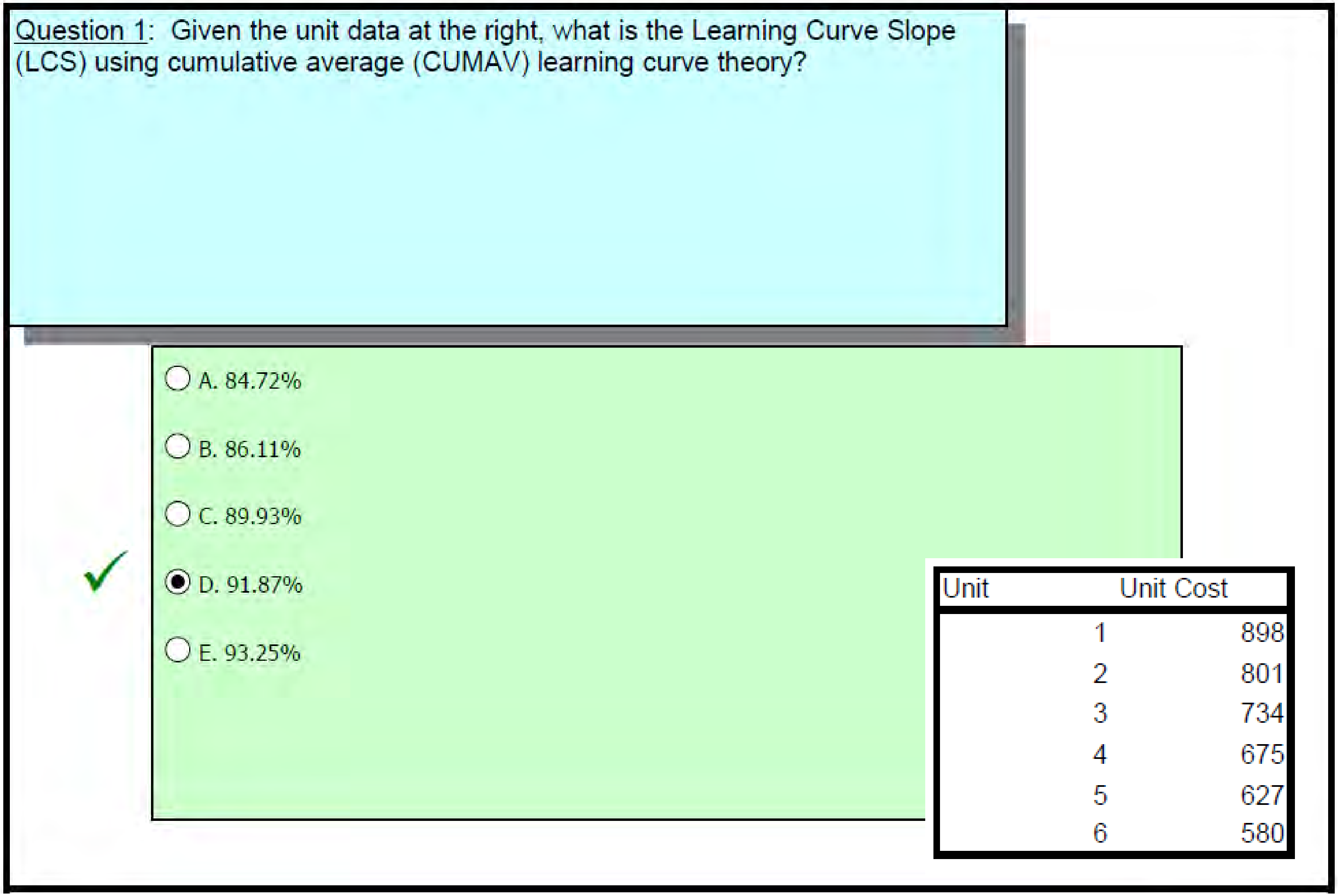
- Learning curves reflect a constant percent decrease in effort for each doubling in quantity
- Two competing learning curve theories:
  - Cumulative Average (CUMAV)
  - Unit Learning Curve (ULC)
  - Smoothing effect of CUMAV gives it an unfair advantage
- Application of learning curves
  - Extrapolation from Actuals for ongoing production run
  - Choice of learning curve (slope) for new program

Sample Test Questions

Our approach to sample test questions:

1. Present the question, the data and the answer up front
2. Review the CEBoK explanation
3. Provide step by step instructions to solve the problems
4. Review takeaways to remember for the exam and life

Question 1



Question 1: Given the unit data at the right, what is the Learning Curve Slope (LCS) using cumulative average (CUMAV) learning curve theory?

○ A. 84.72%

○ B. 86.11%

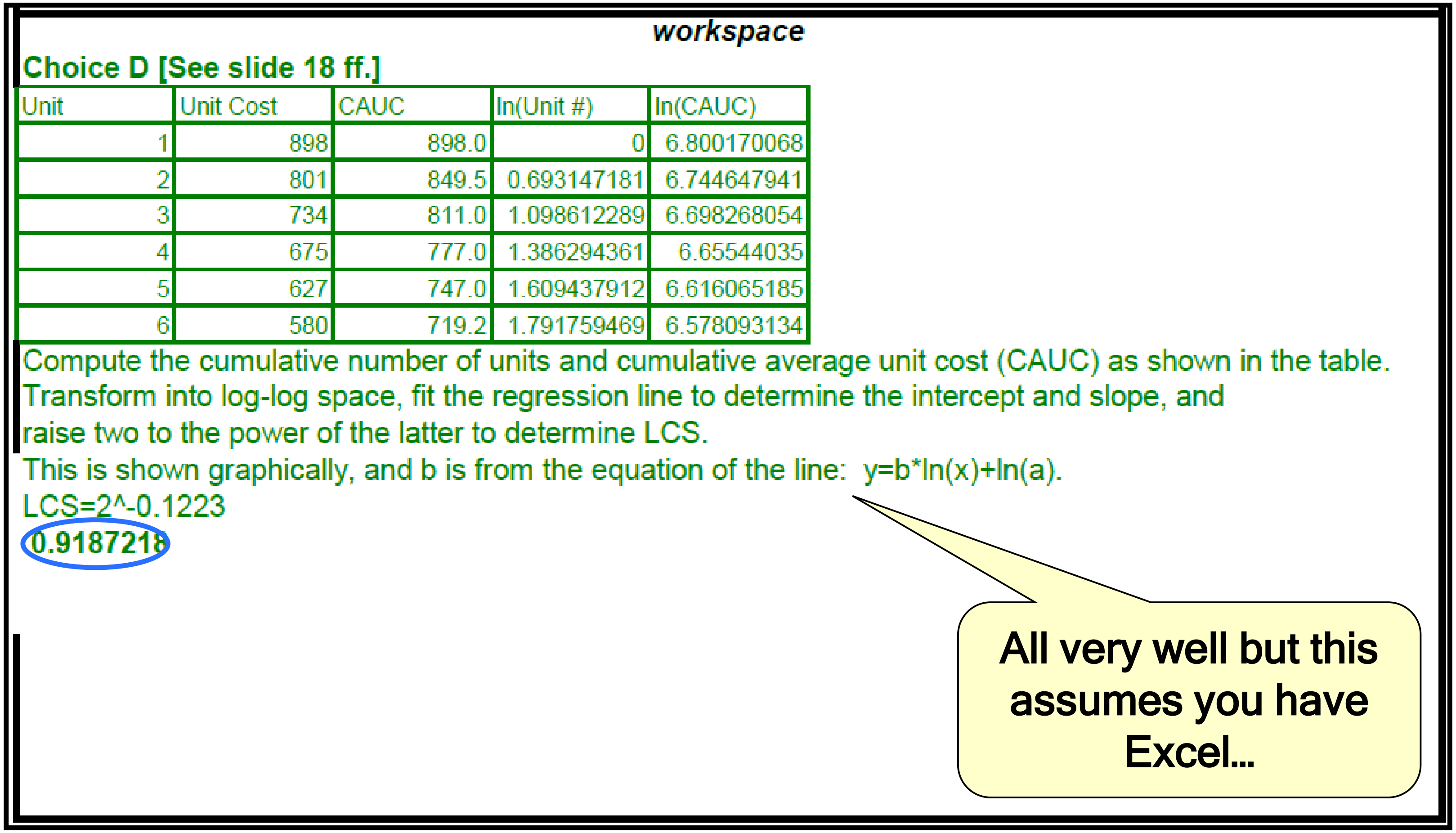
○ C. 89.93%

✓ ● D. 91.87%

○ E. 93.25%

Unit	Unit Cost
1	898
2	801
3	734
4	675
5	627
6	580

Question 1



workspace

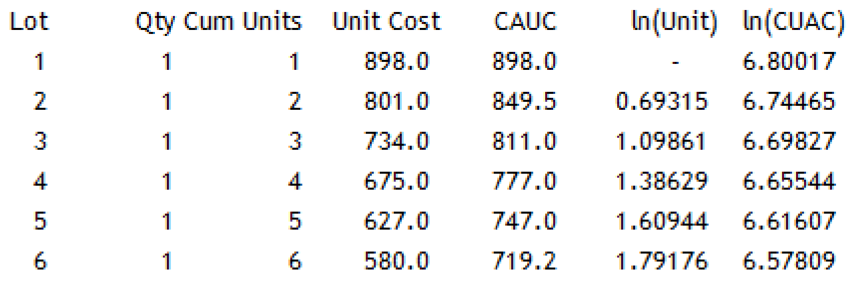
Choice D [See slide 18 ff.]

Unit	Unit Cost	CAUC	ln(Unit #)	ln(CAUC)
1	898	898.0	0	6.800170068
2	801	849.5	0.693147181	6.744647941
3	734	811.0	1.098612289	6.698268054
4	675	777.0	1.386294361	6.65544035
5	627	747.0	1.609437912	6.616065185
6	580	719.2	1.791759469	6.578093134

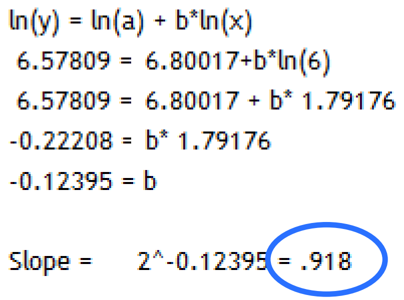
Compute the cumulative number of units and cumulative average unit cost (CAUC) as shown in the table.
Transform into log-log space, fit the regression line to determine the intercept and slope, and
raise two to the power of the latter to determine LCS.
This is shown graphically, and b is from the equation of the line: y=b*ln(x)+ln(a).
LCS=2^-0.1223
0.9187218

All very well but this assumes you have Excel…

Question 1



Lot	Qty	Cum Units	Unit Cost	CAUC	ln(Unit)	ln(CUAC)
1	1	1	898.0	898.0	-	6.80017
2	1	2	801.0	849.5	0.69315	6.74465
3	1	3	734.0	811.0	1.09861	6.69827
4	1	4	675.0	777.0	1.38629	6.65544
5	1	5	627.0	747.0	1.60944	6.61607
6	1	6	580.0	719.2	1.79176	6.57809



ln(y) = ln(a) + b*ln(x)

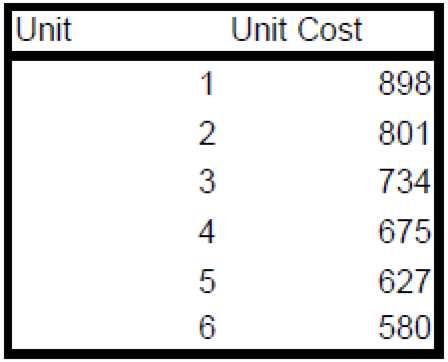
6.57809 = 6.80017+b*ln(6)

6.57809 = 6.80017 + b* 1.79176

-0.22208 = b* 1.79176

-0.12395 = b

Slope = 2^-0.12395 = .918

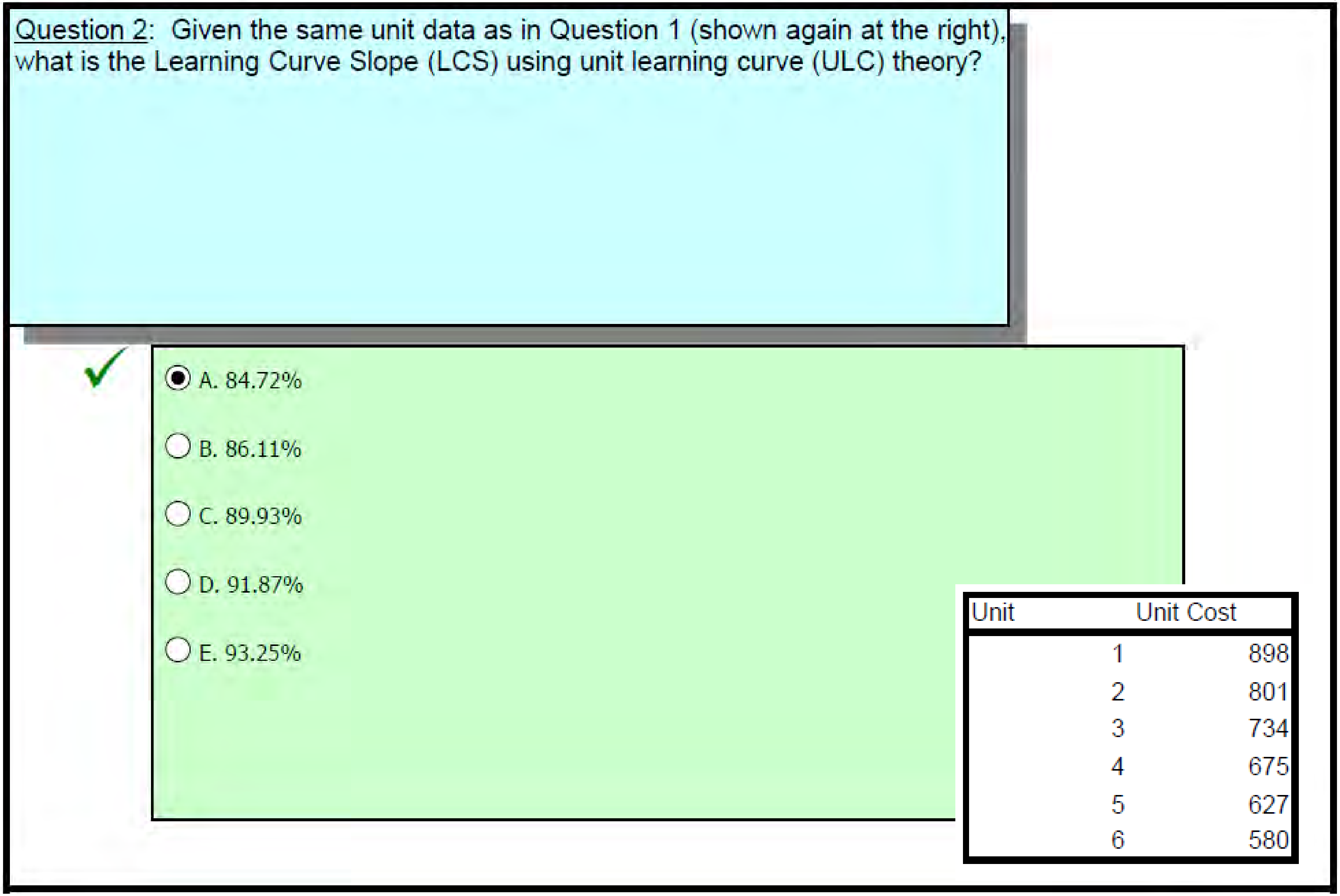


Unit	Unit Cost
1	898
2	801
3	734
4	675
5	627
6	580

Takeaways:

- **Set up the table the same way each time**
- **ln(y) = ln(a) + b*ln(x)**
- **LCS = 2^b**

Question 2



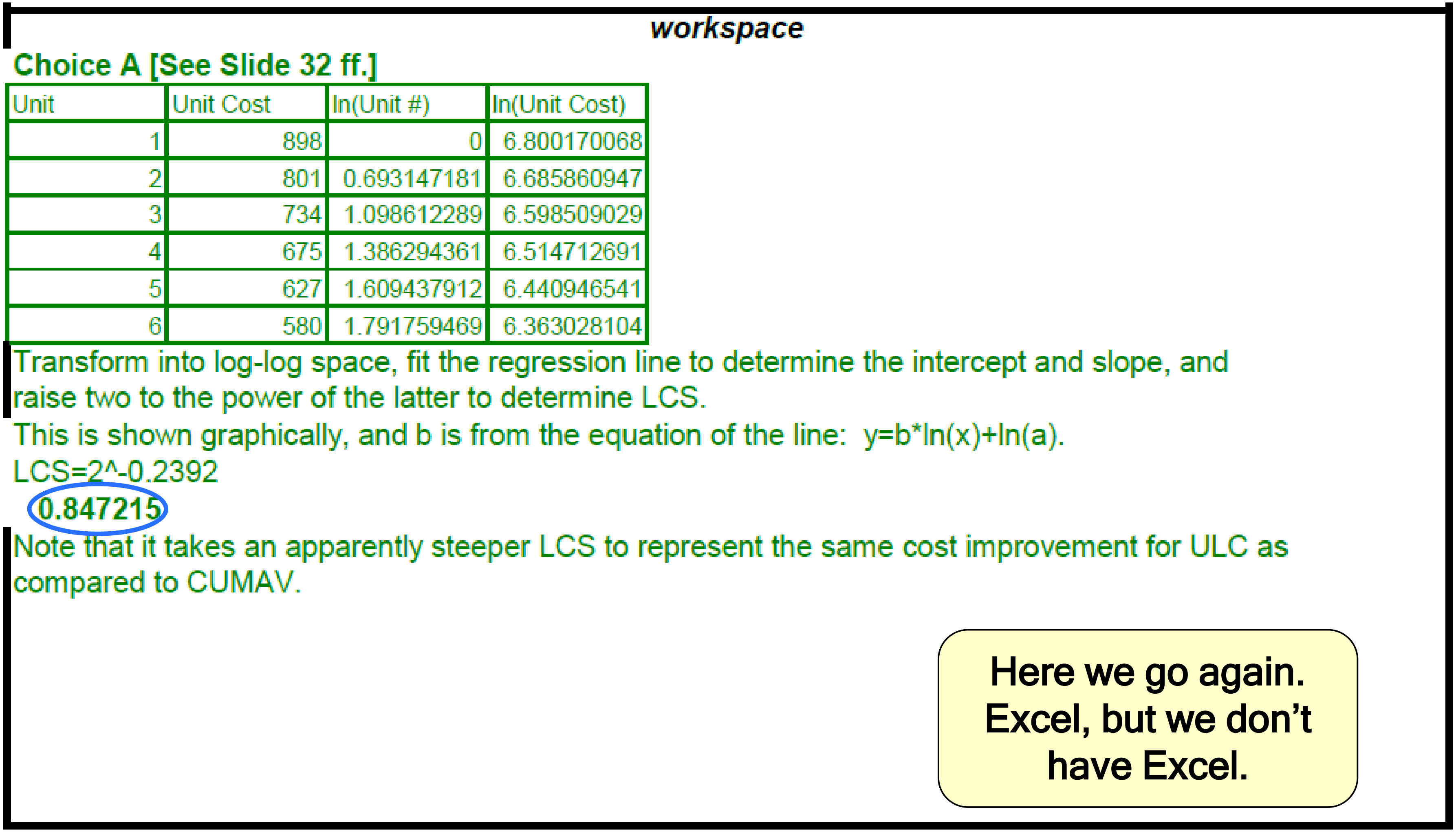
Question 2: Given the same unit data as in Question 1 (shown again at the right), what is the Learning Curve Slope (LCS) using unit learning curve (ULC) theory?

✓

- ● A. 84.72%
- ○ B. 86.11%
- ○ C. 89.93%
- ○ D. 91.87%
- ○ E. 93.25%

Unit	Unit Cost
1	898
2	801
3	734
4	675
5	627
6	580

Question 2



workspace

Choice A [See Slide 32 ff.]

Unit	Unit Cost	ln(Unit #)	ln(Unit Cost)
1	898	0	6.800170068
2	801	0.693147181	6.685860947
3	734	1.098612289	6.598509029
4	675	1.386294361	6.514712691
5	627	1.609437912	6.440946541
6	580	1.791759469	6.363028104

Transform into log-log space, fit the regression line to determine the intercept and slope, and raise two to the power of the latter to determine LCS.

This is shown graphically, and b is from the equation of the line: y=b*ln(x)+ln(a).

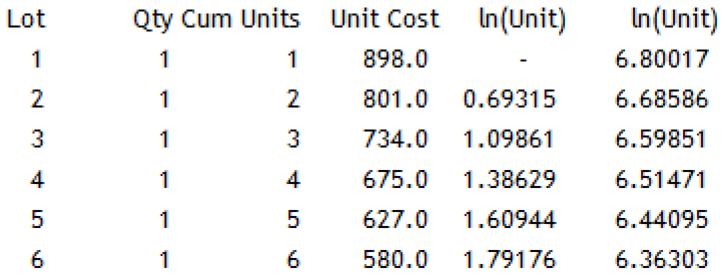
LCS=2^-0.2392

0.847215

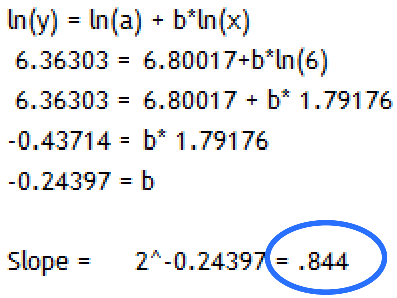
Note that it takes an apparently steeper LCS to represent the same cost improvement for ULC as compared to CUMAV.

Here we go again. Excel, but we don't have Excel.

Question 2



Lot	Qty	Cum Units	Unit Cost	ln(Unit)	ln(Unit)
1	1	1	898.0	-	6.80017
2	1	2	801.0	0.69315	6.68586
3	1	3	734.0	1.09861	6.59851
4	1	4	675.0	1.38629	6.51471
5	1	5	627.0	1.60944	6.44095
6	1	6	580.0	1.79176	6.36303



ln(y) = ln(a) + b*ln(x)

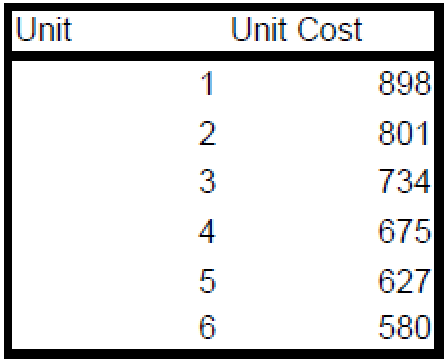
6.36303 = 6.80017+b*ln(6)

6.36303 = 6.80017 + b* 1.79176

-0.43714 = b* 1.79176

-0.24397 = b

Slope = 2^-0.24397 = .844

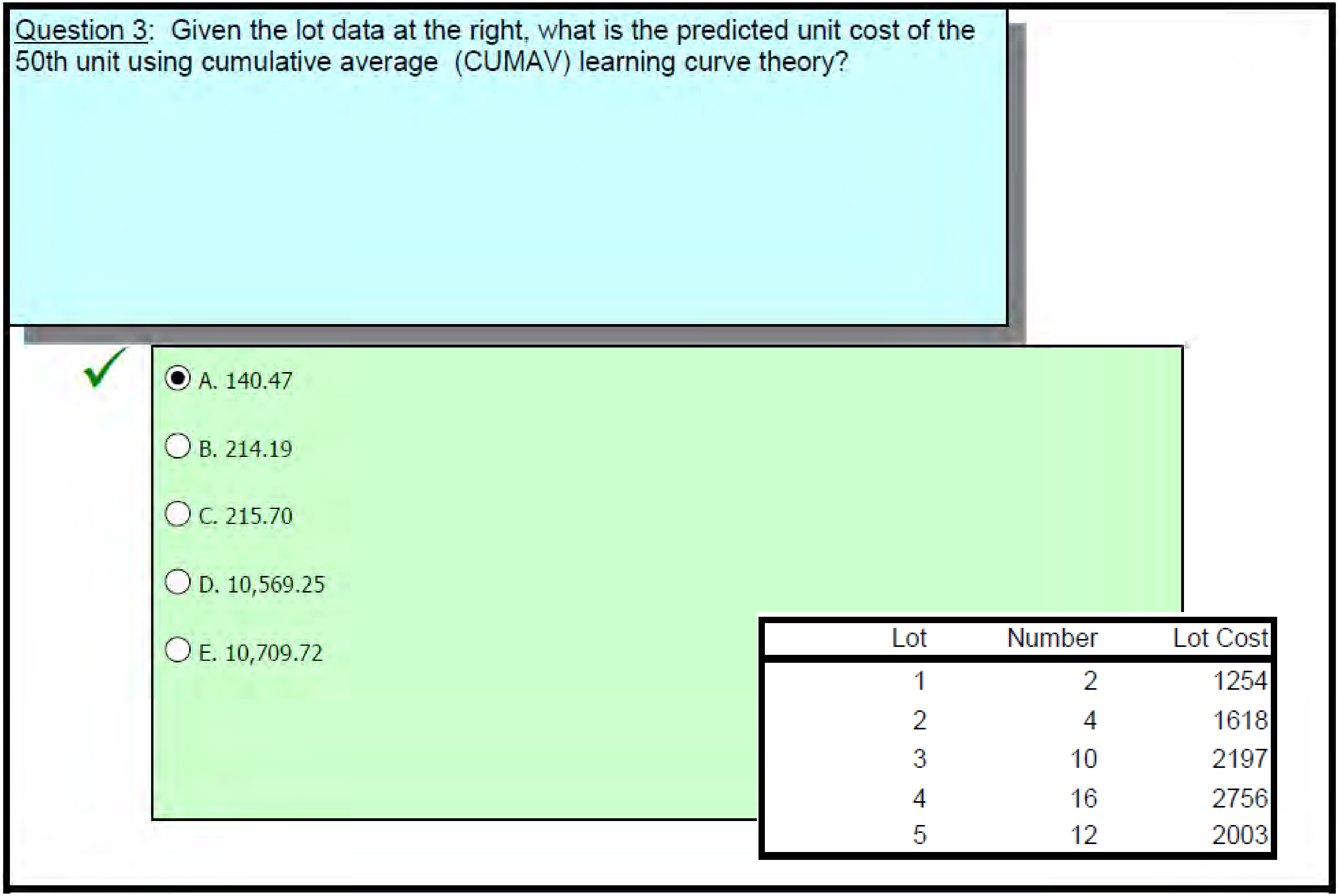


Unit	Unit Cost
1	898
2	801
3	734
4	675
5	627
6	580

Takeaways:

- **Set up the table the same way each time**
- **ln(y) = ln(a) + b*ln(x)**
- **LCS = 2^b**

Question 3



Question 3: Given the lot data at the right, what is the predicted unit cost of the 50th unit using cumulative average (CUMAV) learning curve theory?

✓ ● A. 140.47

○ B. 214.19

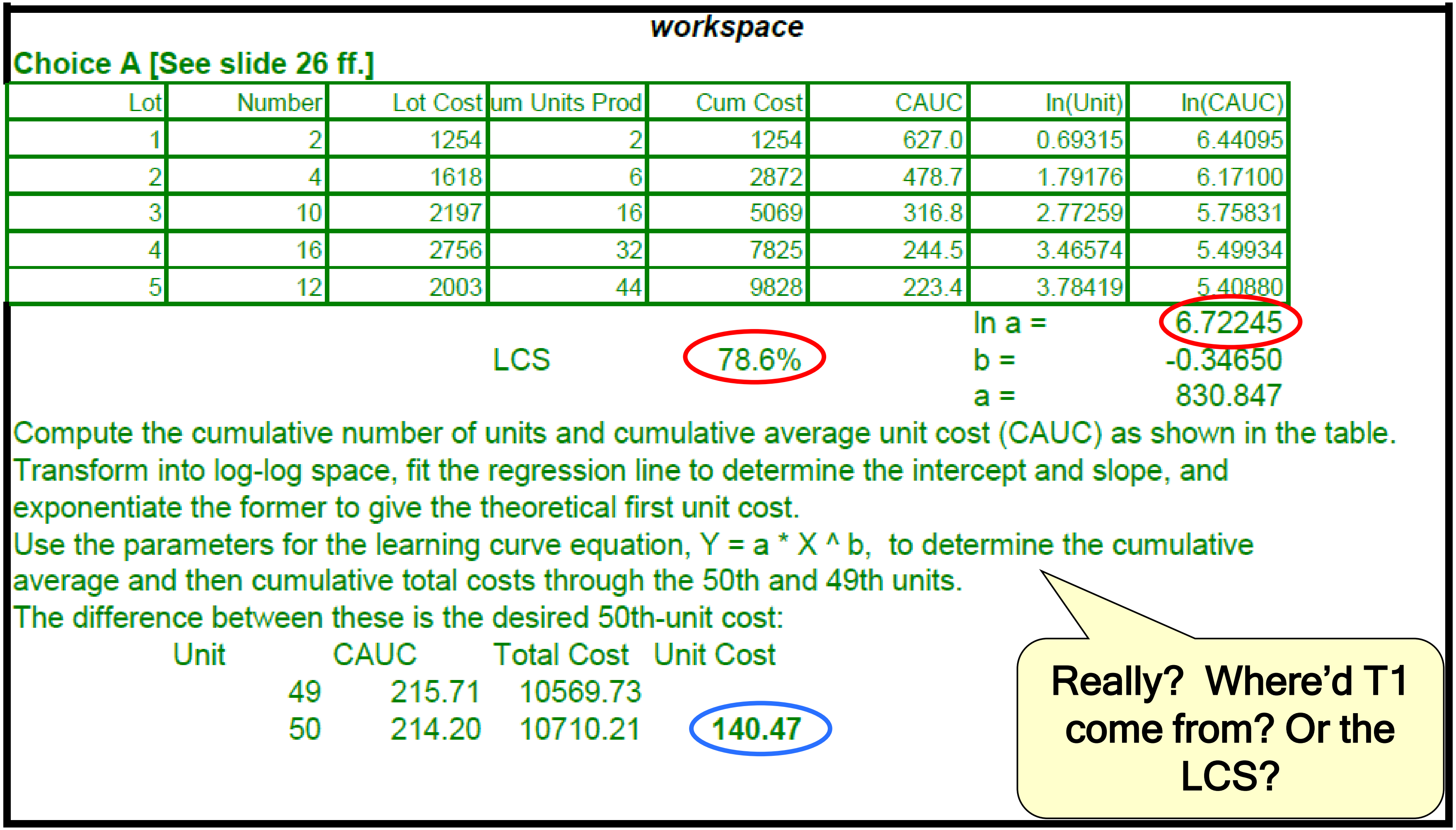
○ C. 215.70

○ D. 10,569.25

○ E. 10,709.72

Lot	Number	Lot Cost
1	2	1254
2	4	1618
3	10	2197
4	16	2756
5	12	2003

Question 3



workspace

Choice A [See slide 26 ff.]

Lot	Number	Lot Cost	um Units Prod	Cum Cost	CAUC	ln(Unit)	ln(CAUC)
1	2	1254	2	1254	627.0	0.69315	6.44095
2	4	1618	6	2872	478.7	1.79176	6.17100
3	10	2197	16	5069	316.8	2.77259	5.75831
4	16	2756	32	7825	244.5	3.46574	5.49934
5	12	2003	44	9828	223.4	3.78419	5.40880

ln a = 6.72245

LCS 78.6%

b = -0.34650

a = 830.847

Compute the cumulative number of units and cumulative average unit cost (CAUC) as shown in the table.
Transform into log-log space, fit the regression line to determine the intercept and slope, and exponentiate the former to give the theoretical first unit cost.
Use the parameters for the learning curve equation, Y = a * X ^ b, to determine the cumulative average and then cumulative total costs through the 50th and 49th units.
The difference between these is the desired 50th-unit cost:

Unit	CAUC	Total Cost	Unit Cost
49	215.71	10569.73	
50	214.20	10710.21	140.47

Really? Where'd T1 come from? Or the LCS?

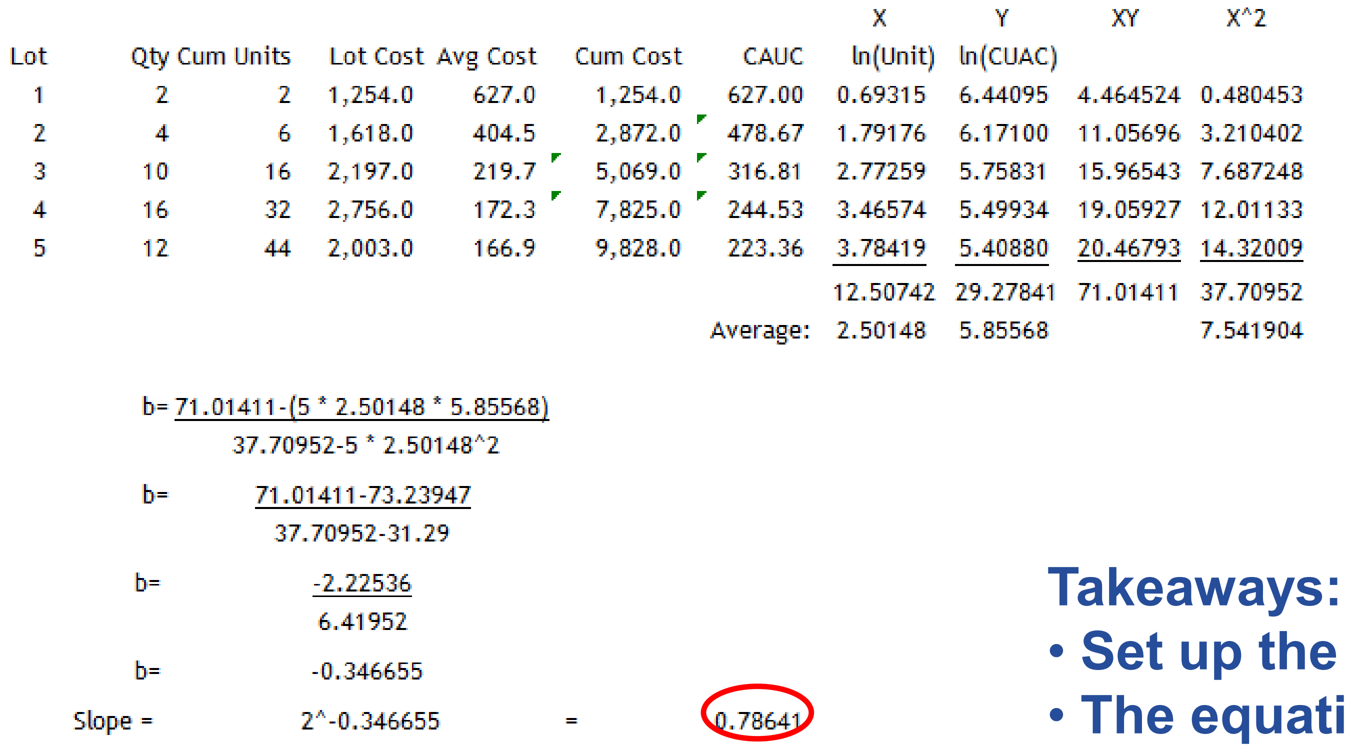
Question 3

Solve for b

$$b = \frac{\sum_{i=1}^{n} X_i Y_i - n\overline{X}\,\overline{Y}}{\sum_{i=1}^{n} X_i^{\,2} - n\overline{X}^{2}}$$

X and Y are the raw values of the 2 variables and
$\overline{X}$ and $\overline{Y}$ are the means of the 2 variables

Lot	Number	Lot Cost
1	2	1254
2	4	1618
3	10	2197
4	16	2756
5	12	2003



							X	Y	XY	X^2
Lot	Qty	Cum Units	Lot Cost	Avg Cost	Cum Cost	CAUC	ln(Unit)	ln(CUAC)		
1	2	2	1,254.0	627.0	1,254.0	627.00	0.69315	6.44095	4.464524	0.480453
2	4	6	1,618.0	404.5	2,872.0	478.67	1.79176	6.17100	11.05696	3.210402
3	10	16	2,197.0	219.7	5,069.0	316.81	2.77259	5.75831	15.96543	7.687248
4	16	32	2,756.0	172.3	7,825.0	244.53	3.46574	5.49934	19.05927	12.01133
5	12	44	2,003.0	166.9	9,828.0	223.36	3.78419	5.40880	20.46793	14.32009
							12.50742	29.27841	71.01411	37.70952
						Average:	2.50148	5.85568		7.541904

b= 71.01411-(5 * 2.50148 * 5.85568)
37.70952-5 * 2.50148^2

b= 71.01411-73.23947
37.70952-31.29

b= -2.22536
6.41952

b= -0.346655

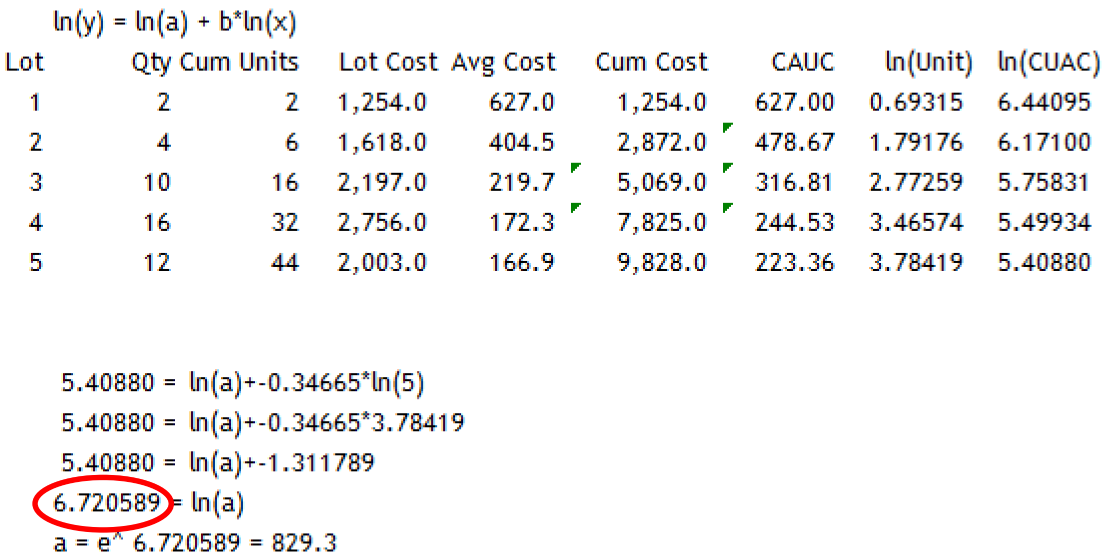
Slope = 2^-0.346655 = 0.78641

Takeaways:

- **Set up the table the same…**
- **The equation for b (really)**

Question 3

Solve for a



ln(y) = ln(a) + b*ln(x)

Lot	Qty	Cum Units	Lot Cost	Avg Cost	Cum Cost	CAUC	ln(Unit)	ln(CUAC)
1	2	2	1,254.0	627.0	1,254.0	627.00	0.69315	6.44095
2	4	6	1,618.0	404.5	2,872.0	478.67	1.79176	6.17100
3	10	16	2,197.0	219.7	5,069.0	316.81	2.77259	5.75831
4	16	32	2,756.0	172.3	7,825.0	244.53	3.46574	5.49934
5	12	44	2,003.0	166.9	9,828.0	223.36	3.78419	5.40880

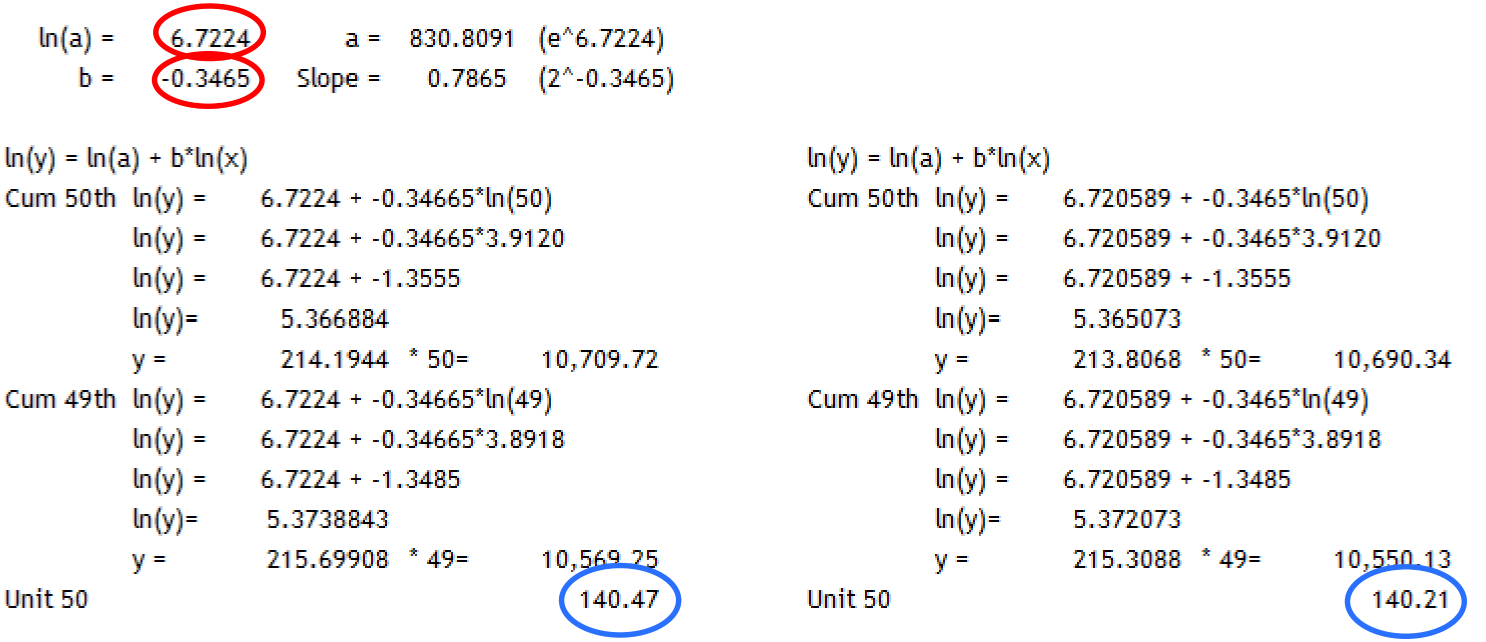
5.40880 = ln(a)+-0.34665*ln(5)
5.40880 = ln(a)+-0.34665*3.78419
5.40880 = ln(a)+-1.311789
6.720589 = ln(a)
a = e^ 6.720589 = 829.3

Takeaways:

- **Set up the table the same way each time**
- **ln(y) = ln(a) + b*ln(x)**
- **To convert ln(a) to a, raise e^ln(a)**
- **Know your calculator before the test**

Question 3

Solve the question



```
ln(a) =   6.7224        a =   830.8091   (e^6.7224)
    b =  -0.3465    Slope =     0.7865   (2^-0.3465)

ln(y) = ln(a) + b*ln(x)                                      ln(y) = ln(a) + b*ln(x)
Cum 50th  ln(y) =   6.7224 + -0.34665*ln(50)                 Cum 50th  ln(y) =   6.720589 + -0.3465*ln(50)
          ln(y) =   6.7224 + -0.34665*3.9120                           ln(y) =   6.720589 + -0.3465*3.9120
          ln(y) =   6.7224 + -1.3555                                   ln(y) =   6.720589 + -1.3555
          ln(y)=      5.366884                                         ln(y)=      5.365073
          y =         214.1944  * 50=     10,709.72                    y =         213.8068  * 50=     10,690.34
Cum 49th  ln(y) =   6.7224 + -0.34665*ln(49)                 Cum 49th  ln(y) =   6.720589 + -0.3465*ln(49)
          ln(y) =   6.7224 + -0.34665*3.8918                           ln(y) =   6.720589 + -0.3465*3.8918
          ln(y) =   6.7224 + -1.3485                                   ln(y) =   6.720589 + -1.3485
          ln(y)=    5.3738843                                          ln(y)=      5.372073
          y =       215.69908  * 49=      10,569.25                    y =         215.3088  * 49=     10,550.13
Unit 50                                    140.47            Unit 50                                    140.21
```

Takeaways:

- **ln(y) = ln(a) + b*ln(x)**
- **LCS = 2^b**
- **Use care on CUAC problems – do they want the CUAC or the unit?**

Question 4

Question 4: Given the same lot data as in Question 3 (shown again at the right), what is the cumulative average cost of the first 50 units predict by cumulative average cost (CUMAV) learning curve theory?

○ A. 140.47

✓ ● B. 214.19

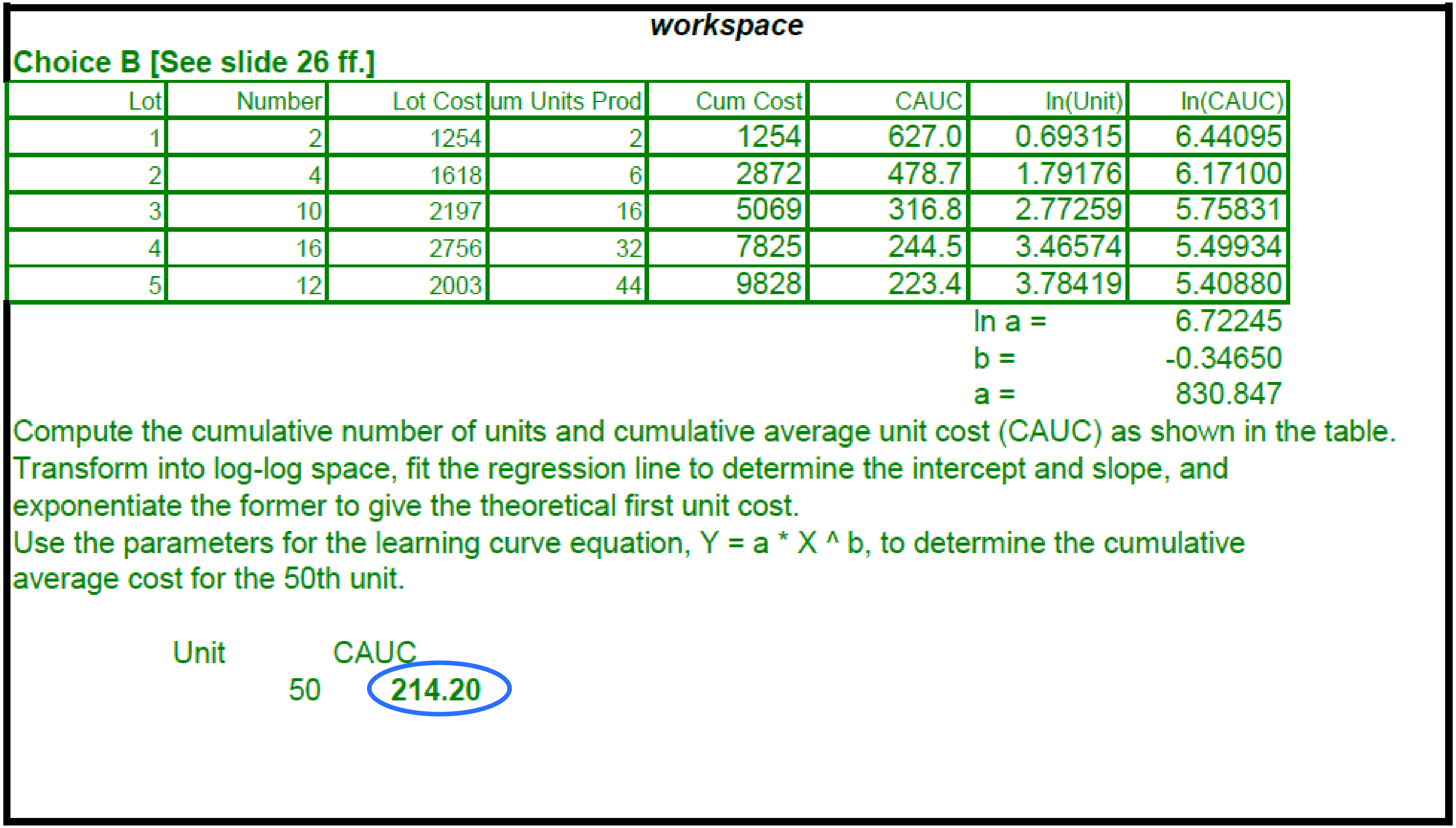
○ C. 215.70

○ D. 10,569.25

○ E. 10,709.72

Lot	Number	Lot Cost
1	2	1254
2	4	1618
3	10	2197
4	16	2756
5	12	2003

Question 4



workspace

Choice B [See slide 26 ff.]

Lot	Number	Lot Cost	um Units Prod	Cum Cost	CAUC	ln(Unit)	ln(CAUC)
1	2	1254	2	1254	627.0	0.69315	6.44095
2	4	1618	6	2872	478.7	1.79176	6.17100
3	10	2197	16	5069	316.8	2.77259	5.75831
4	16	2756	32	7825	244.5	3.46574	5.49934
5	12	2003	44	9828	223.4	3.78419	5.40880

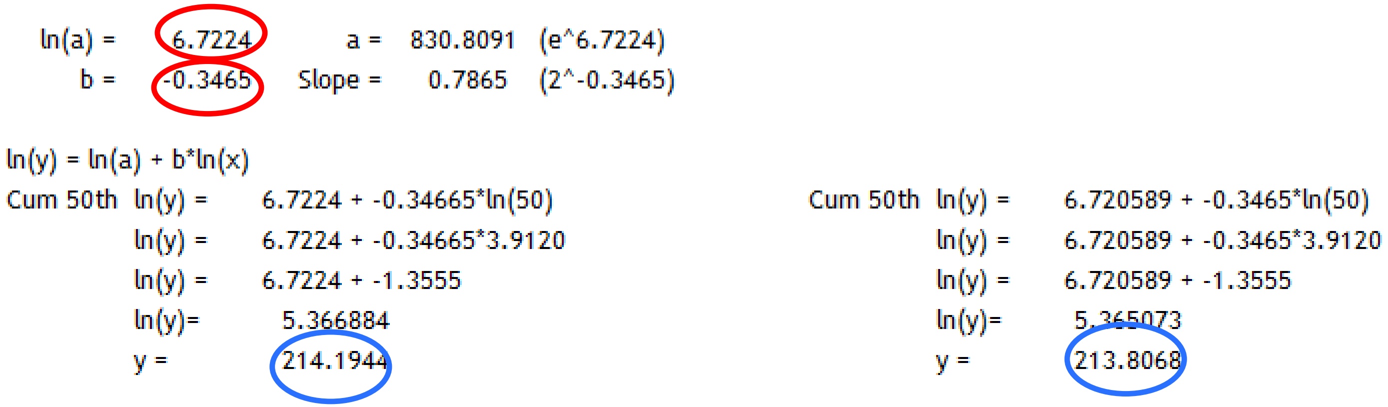
ln a = 6.72245
b = -0.34650
a = 830.847

Compute the cumulative number of units and cumulative average unit cost (CAUC) as shown in the table. Transform into log-log space, fit the regression line to determine the intercept and slope, and exponentiate the former to give the theoretical first unit cost.
Use the parameters for the learning curve equation, Y = a * X ^ b, to determine the cumulative average cost for the 50th unit.

Unit	CAUC
50	**214.20**

Question 4

Solve the question



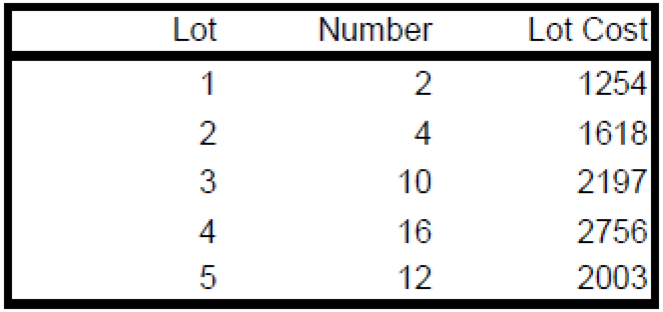
ln(a) = 6.7224 a = 830.8091 (e^6.7224)

b = -0.3465 Slope = 0.7865 (2^-0.3465)

ln(y) = ln(a) + b*ln(x)

Cum 50th ln(y) = 6.7224 + -0.34665*ln(50)
ln(y) = 6.7224 + -0.34665*3.9120
ln(y) = 6.7224 + -1.3555
ln(y)= 5.366884
y = 214.1944

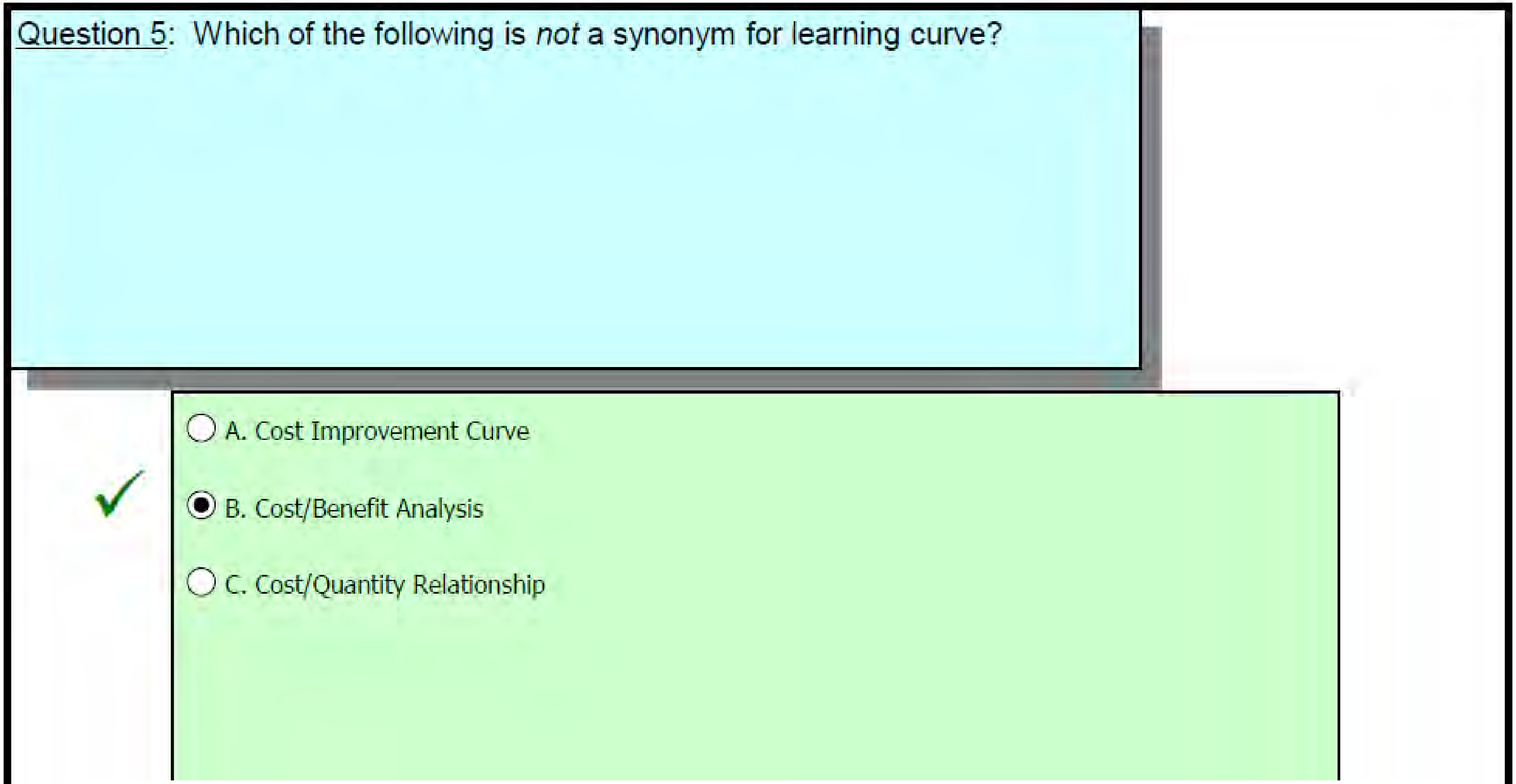
Cum 50th ln(y) = 6.720589 + -0.3465*ln(50)
ln(y) = 6.720589 + -0.3465*3.9120
ln(y) = 6.720589 + -1.3555
ln(y)= 5.365073
y = 213.8068



Lot	Number	Lot Cost
1	2	1254
2	4	1618
3	10	2197
4	16	2756
5	12	2003

Takeaways:
- ln(y) = ln(a) + b*ln(x)
- To convert ln(y) to y, raise e^ln(y)

Question 5



Question 5: Which of the following is *not* a synonym for learning curve?

○ A. Cost Improvement Curve

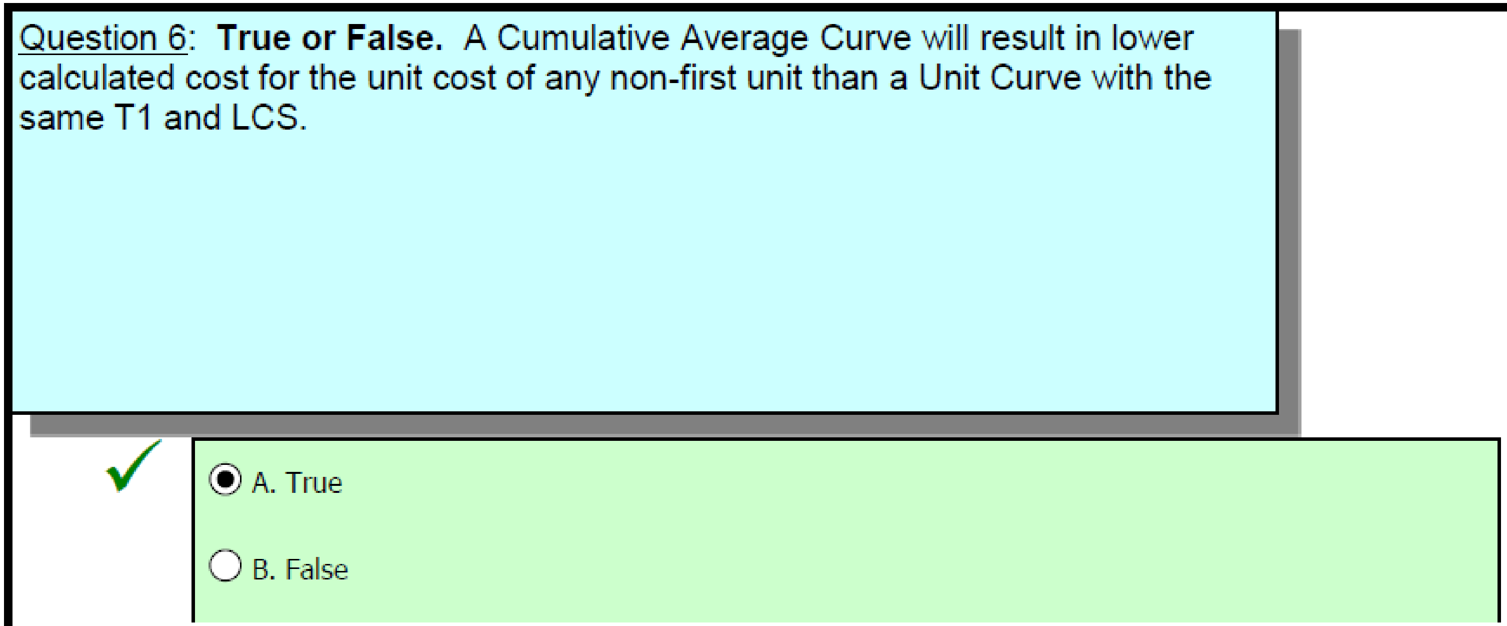
✓ ● B. Cost/Benefit Analysis

○ C. Cost/Quantity Relationship

Choice B [See slide 7]

A learning curve is a cost/quantity relationship, also referred to as a cost improvement curve (CIC). **Cost/Benefit Analysis is a discipline within the broad scope of cost estimating,** discussed in Module 13 Economic Analysis.

Question 6



Question 6: **True or False.** A Cumulative Average Curve will result in lower calculated cost for the unit cost of any non-first unit than a Unit Curve with the same T1 and LCS.

✓

● A. True

○ B. False

Choice A [See slide 33]

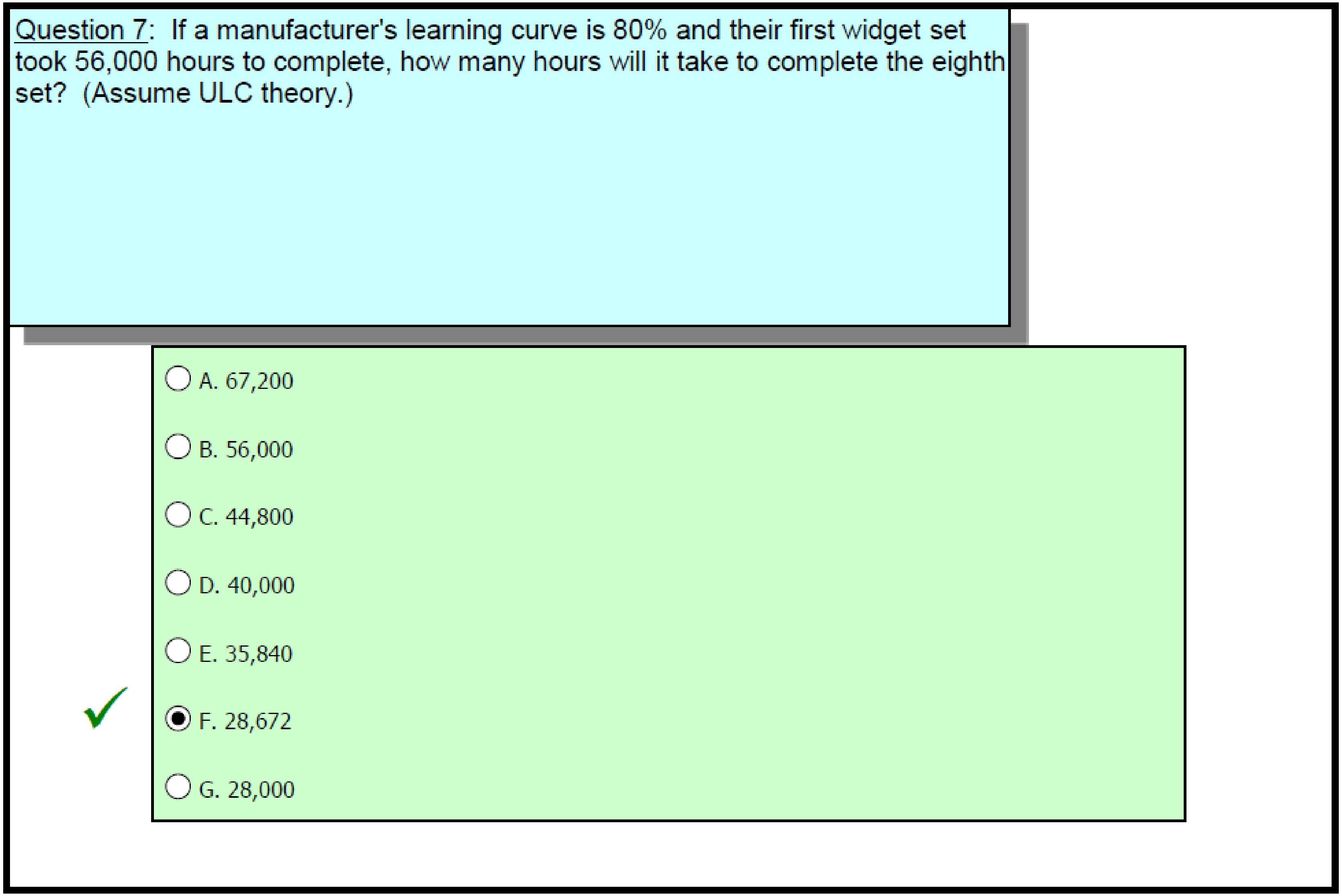
For a cumulative average learning curve and a unit learning curve with the same numerical slope, **the unit cost that follows the cumulative average slope will be lower than the unit cost that follows the unit slope**. Test this comparing the unit cost of the second unit if the T1 is 100 and the LCS is 80% for for both CUMAV and Unit Theories.

T1	LCS
100	80%

Unit Theory	Unit Cost
1	100
2	80

CUMAV	CAUC	Unit cost
1	100	100
2	80	60

Question 7



Question 7: If a manufacturer's learning curve is 80% and their first widget set took 56,000 hours to complete, how many hours will it take to complete the eighth set? (Assume ULC theory.)

○ A. 67,200

○ B. 56,000

○ C. 44,800

○ D. 40,000

○ E. 35,840

✓ ● F. 28,672

○ G. 28,000

Question 7

workspace

Choice F [See slide 31]

Using Unit Learning Curve theory, Y=A*X^b. Y=56,000*8^(ln(80%)/ln(2))

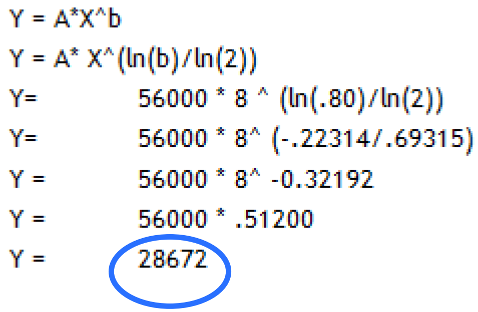
56000*8^(ln(.80)/ln(2)) = **28672**

The easier solution, which could be crucial in saving time on the certification exam, is to notice that going from the 1st unit to the 8th unit represents doubling thrice, so that by the definition of learning curve, the first-unit hours will be multiplied by the LCS thrice:

56000*0.8*0.8*0.8= **28672**

Because in real life it's always that neat. Right?

Question 7

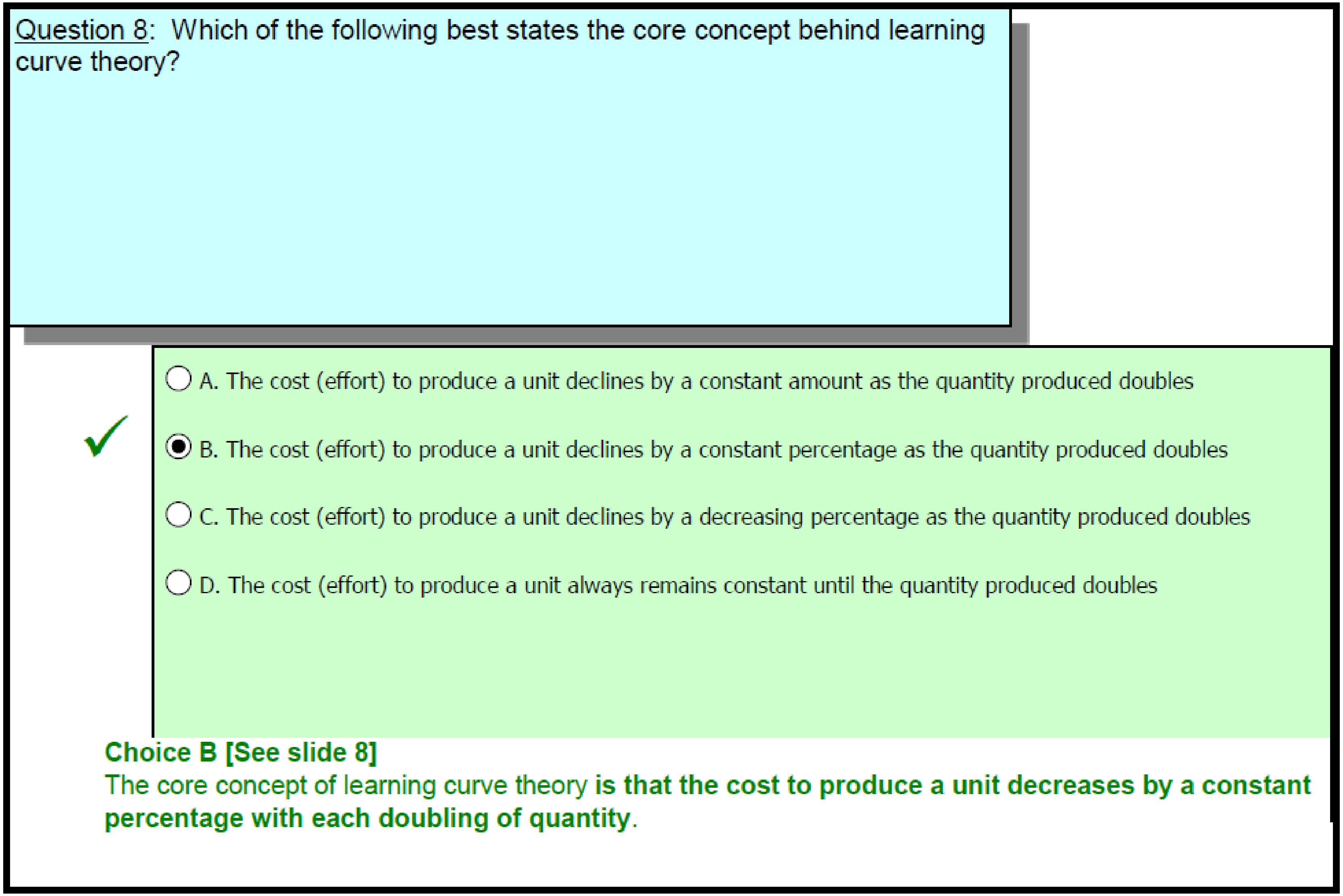


Y = A*X^b
Y = A* X^(ln(b)/ln(2))
Y= 56000 * 8 ^ (ln(.80)/ln(2))
Y= 56000 * 8^ (-.22314/.69315)
Y = 56000 * 8^ -0.32192
Y = 56000 * .51200
Y = 28672

Takeaways:

- **Y= a*X^b**
- **To convert LCS to b: (ln(LCS)/ln(2))**

Question 8



Question 8: Which of the following best states the core concept behind learning curve theory?

○ A. The cost (effort) to produce a unit declines by a constant amount as the quantity produced doubles

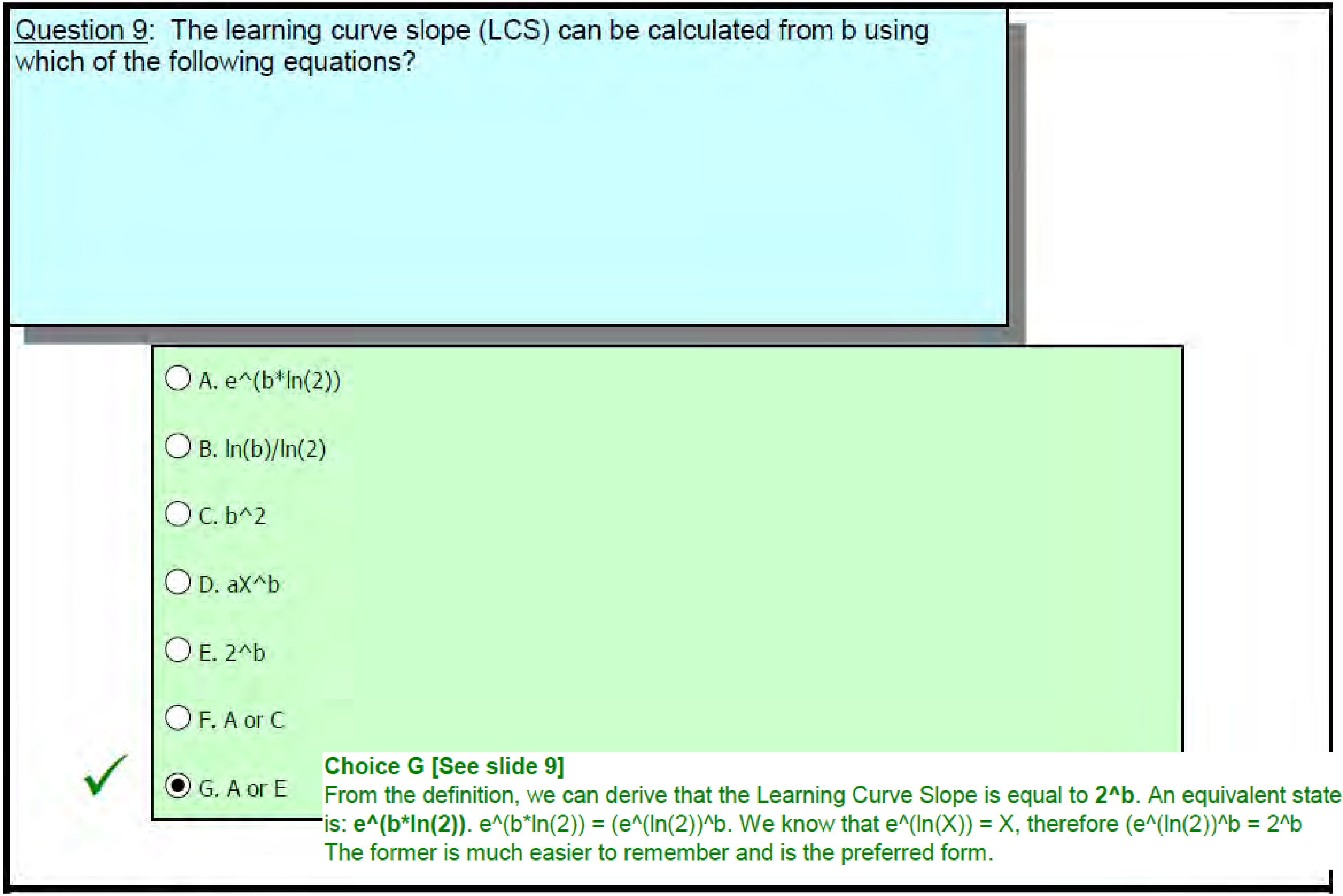
✓ ● B. The cost (effort) to produce a unit declines by a constant percentage as the quantity produced doubles

○ C. The cost (effort) to produce a unit declines by a decreasing percentage as the quantity produced doubles

○ D. The cost (effort) to produce a unit always remains constant until the quantity produced doubles

Choice B [See slide 8]
The core concept of learning curve theory **is that the cost to produce a unit decreases by a constant percentage with each doubling of quantity**.

Question 9



Question 9: The learning curve slope (LCS) can be calculated from b using which of the following equations?

○ A. e^(b*ln(2))

○ B. ln(b)/ln(2)

○ C. b^2

○ D. aX^b

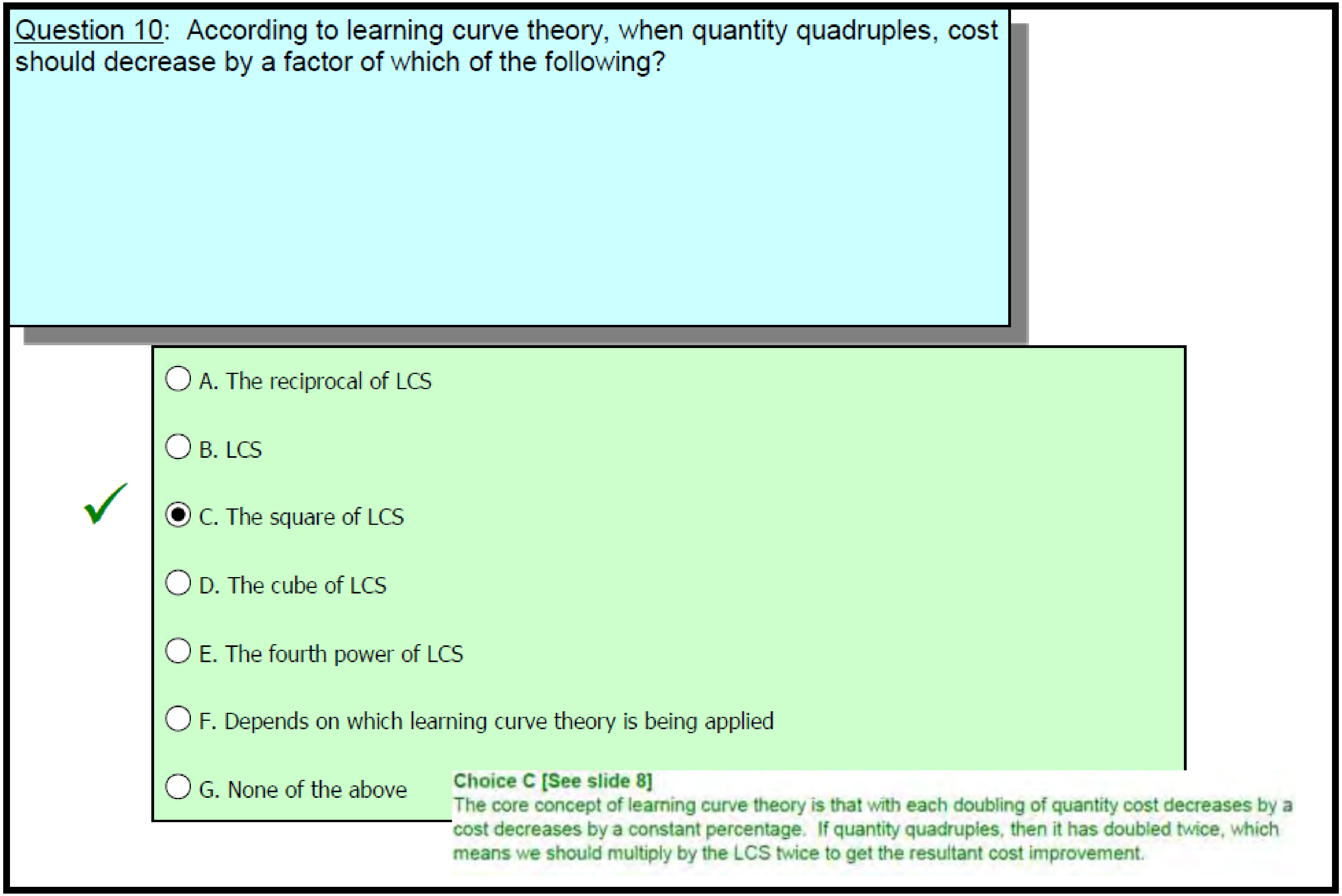
○ E. 2^b

○ F. A or C

✓ ● G. A or E

Choice G [See slide 9]
From the definition, we can derive that the Learning Curve Slope is equal to **2^b**. An equivalent statement is: **e^(b*ln(2))**. e^(b*ln(2)) = (e^(ln(2))^b. We know that e^(ln(X)) = X, therefore (e^(ln(2))^b = 2^b
The former is much easier to remember and is the preferred form.

Question 10



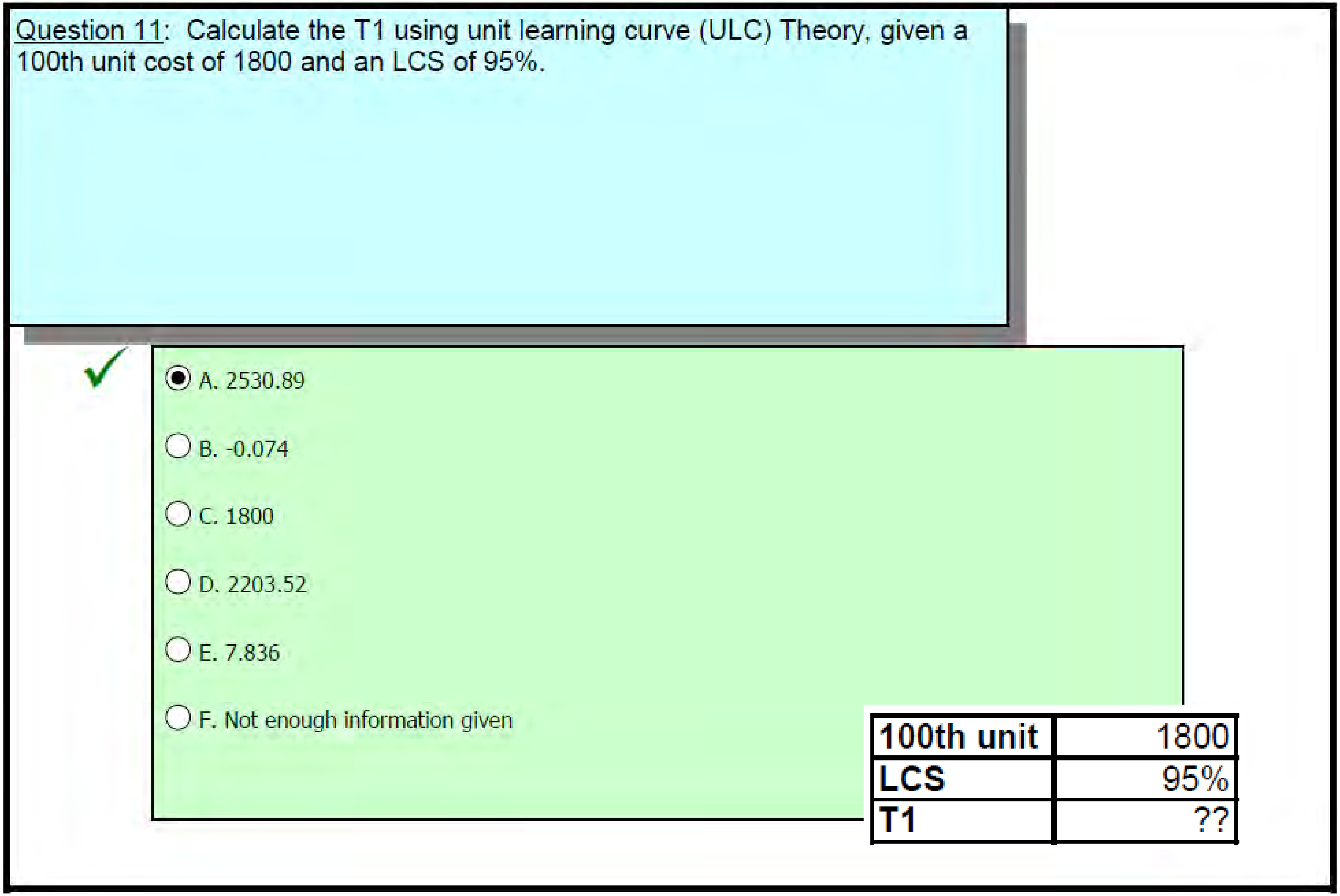
Question 10: According to learning curve theory, when quantity quadruples, cost should decrease by a factor of which of the following?

- ○ A. The reciprocal of LCS
- ○ B. LCS
- ✓ ● C. The square of LCS
- ○ D. The cube of LCS
- ○ E. The fourth power of LCS
- ○ F. Depends on which learning curve theory is being applied
- ○ G. None of the above

Choice C [See slide 8]

The core concept of learning curve theory is that with each doubling of quantity cost decreases by a cost decreases by a constant percentage. If quantity quadruples, then it has doubled twice, which means we should multiply by the LCS twice to get the resultant cost improvement.

Question 11



Question 11: Calculate the T1 using unit learning curve (ULC) Theory, given a 100th unit cost of 1800 and an LCS of 95%.

✓

● A. 2530.89

○ B. -0.074

○ C. 1800

○ D. 2203.52

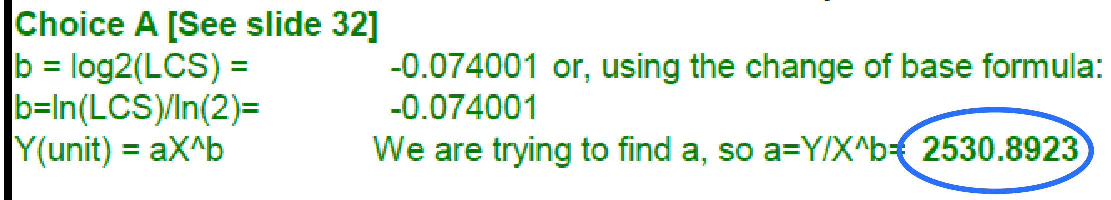
○ E. 7.836

○ F. Not enough information given

100th unit	1800
LCS	95%
T1	??

Question 11

workspace



Choice A [See slide 32]

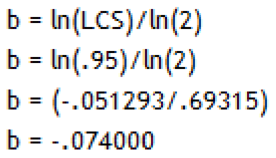
b = log2(LCS) = -0.074001 or, using the change of base formula:

b=ln(LCS)/ln(2)= -0.074001

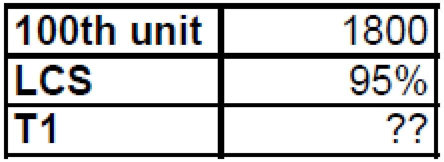
Y(unit) = aX^b We are trying to find a, so a=Y/X^b= **2530.8923**

Question 11

Solve for b

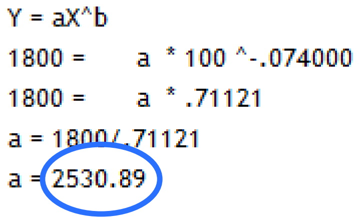


b = ln(LCS)/ln(2)
b = ln(.95)/ln(2)
b = (-.051293/.69315)
b = -.074000



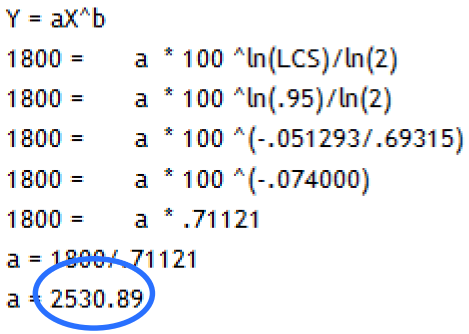
100th unit	1800
LCS	95%
T1	??

Solve for a



Y = aX^b
1800 = a * 100 ^-.074000
1800 = a * .71121
a = 1800/.71121
a = 2530.89

Solve all at once

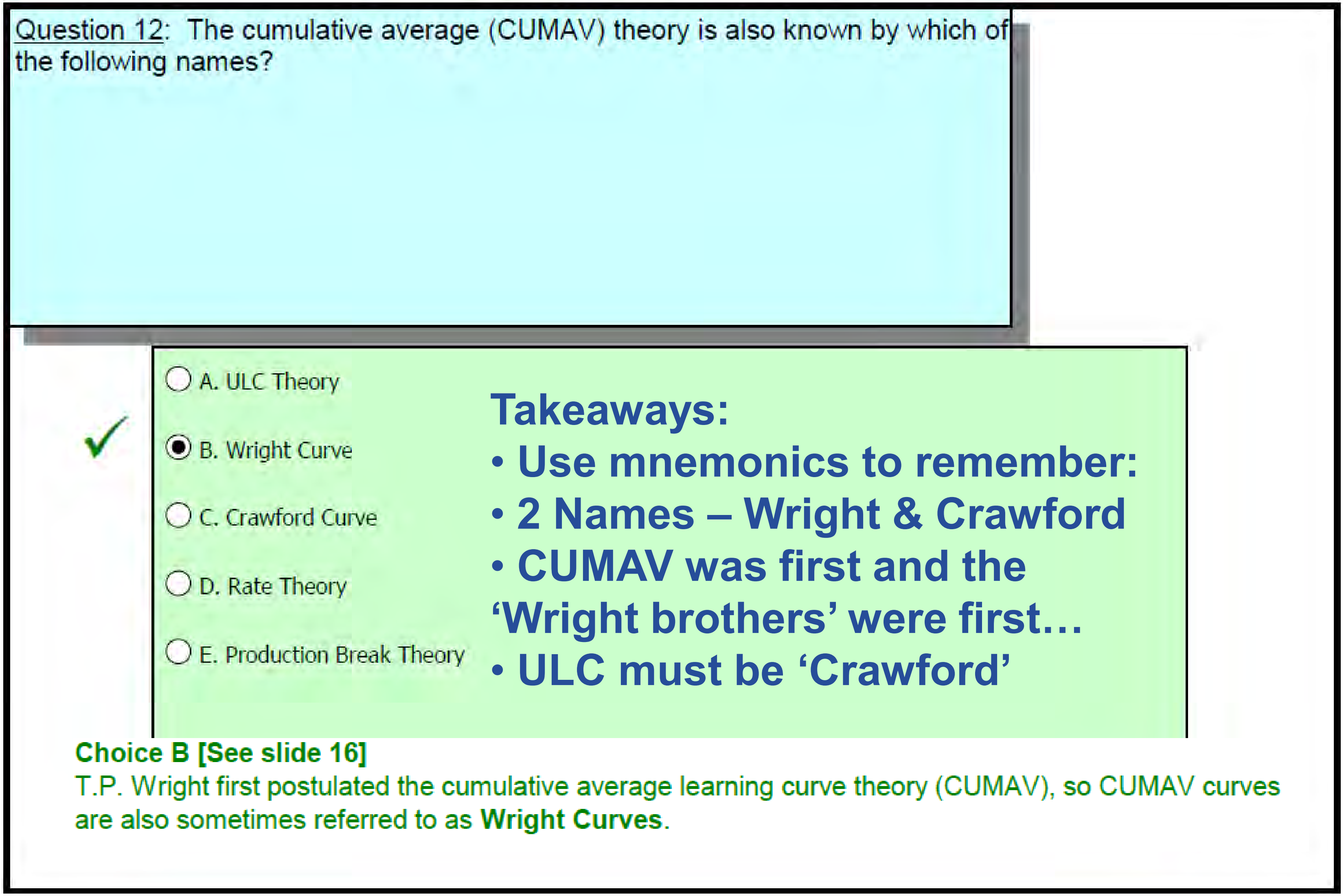


Y = aX^b
1800 = a * 100 ^ln(LCS)/ln(2)
1800 = a * 100 ^ln(.95)/ln(2)
1800 = a * 100 ^(-.051293/.69315)
1800 = a * 100 ^(-.074000)
1800 = a * .71121
a = 1800/.71121
a = 2530.89

Takeaways:

- To convert LCS to b: (ln(LCS)/ln(2))
- Y= a*X^b

Question 12



Question 12: The cumulative average (CUMAV) theory is also known by which of the following names?

○ A. ULC Theory

✓ ● B. Wright Curve

○ C. Crawford Curve

○ D. Rate Theory

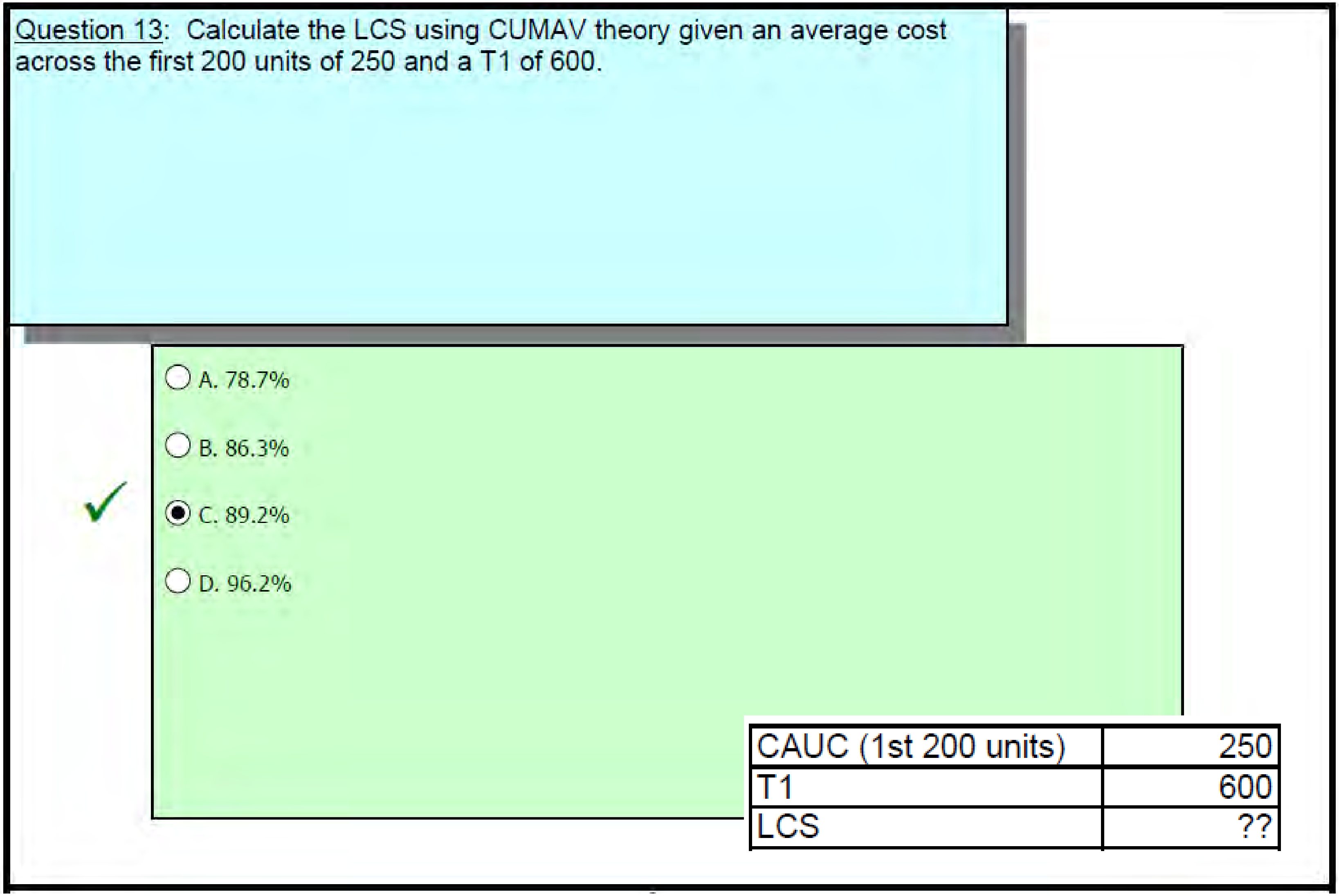
○ E. Production Break Theory

Takeaways:
- **Use mnemonics to remember:**
- **2 Names – Wright & Crawford**
- **CUMAV was first and the 'Wright brothers' were first…**
- **ULC must be 'Crawford'**

Choice B [See slide 16]

T.P. Wright first postulated the cumulative average learning curve theory (CUMAV), so CUMAV curves are also sometimes referred to as **Wright Curves**.

Question 13



Question 13: Calculate the LCS using CUMAV theory given an average cost across the first 200 units of 250 and a T1 of 600.

○ A. 78.7%

○ B. 86.3%

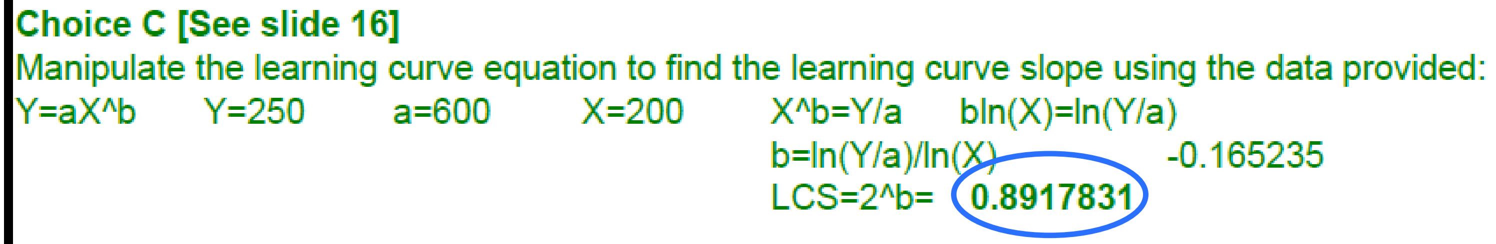
✓ ● C. 89.2%

○ D. 96.2%

CAUC (1st 200 units)	250
T1	600
LCS	??

Question 13

workspace



Choice C [See slide 16]

Manipulate the learning curve equation to find the learning curve slope using the data provided:

Y=aX^b	Y=250	a=600	X=200	X^b=Y/a	bln(X)=ln(Y/a)	
				b=ln(Y/a)/ln(X)		-0.165235
				LCS=2^b=	**0.8917831**	

Question 13

Solve for b

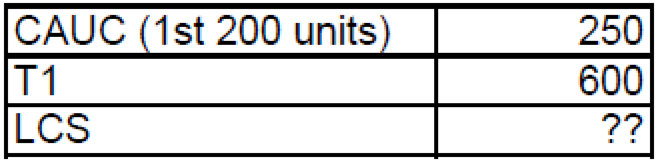
b = ln(Y/a)/ln(X)

b = (ln(250/600)/ln(200))

b = (ln(.416667)/ln(200)

b = (-.87547/5.29832)

b = -.165235



CAUC (1st 200 units)	250
T1	600
LCS	??

Solve for LCS

LCS = 2^b

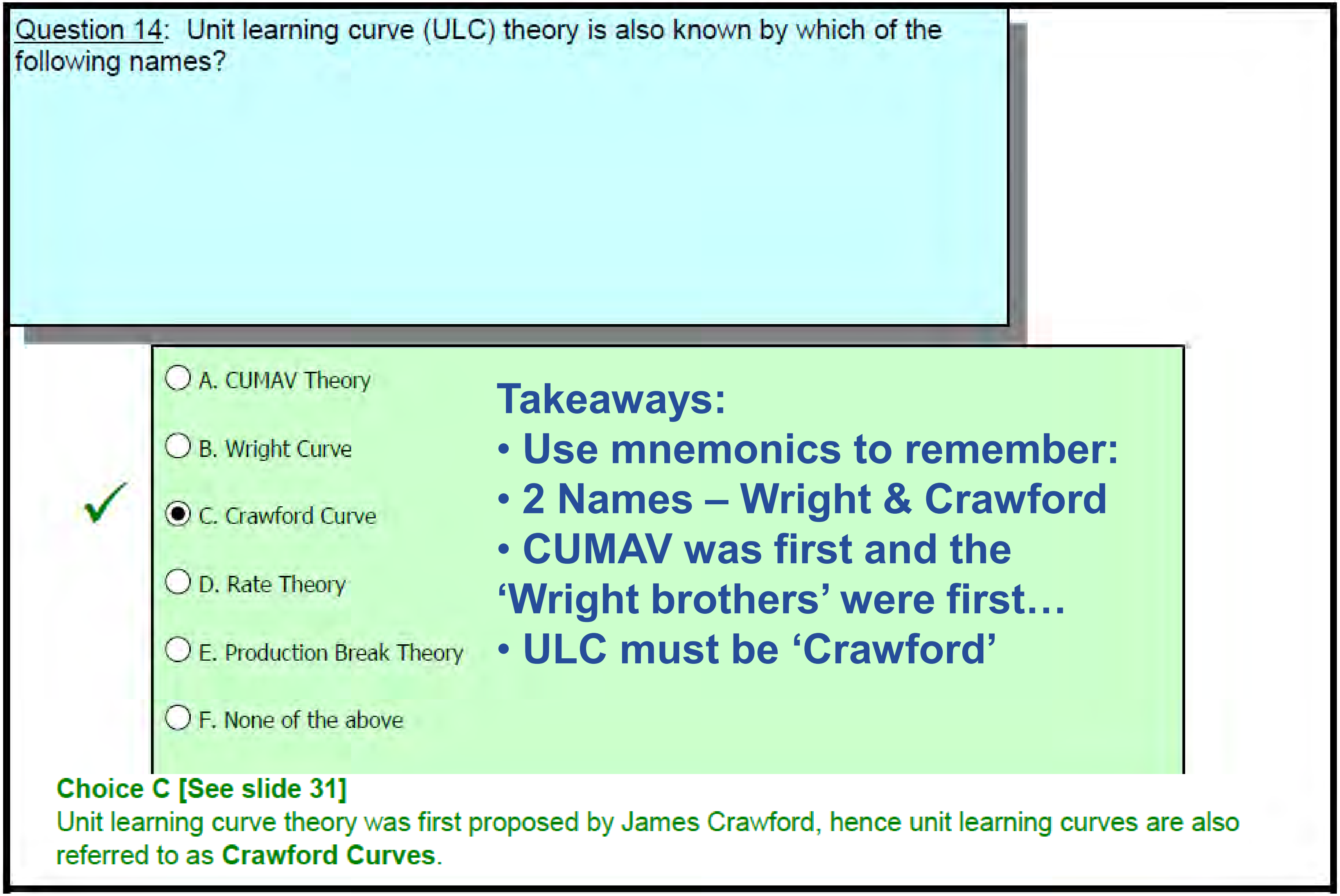
LCS = 2^-.165235

LCS = .891783

Takeaways:

- **ln(y) = ln(a) + b*ln(x)**
- **LCS = 2^b**

Question 14



Question 14: Unit learning curve (ULC) theory is also known by which of the following names?

○ A. CUMAV Theory

○ B. Wright Curve

✓ ● C. Crawford Curve

○ D. Rate Theory

○ E. Production Break Theory

○ F. None of the above

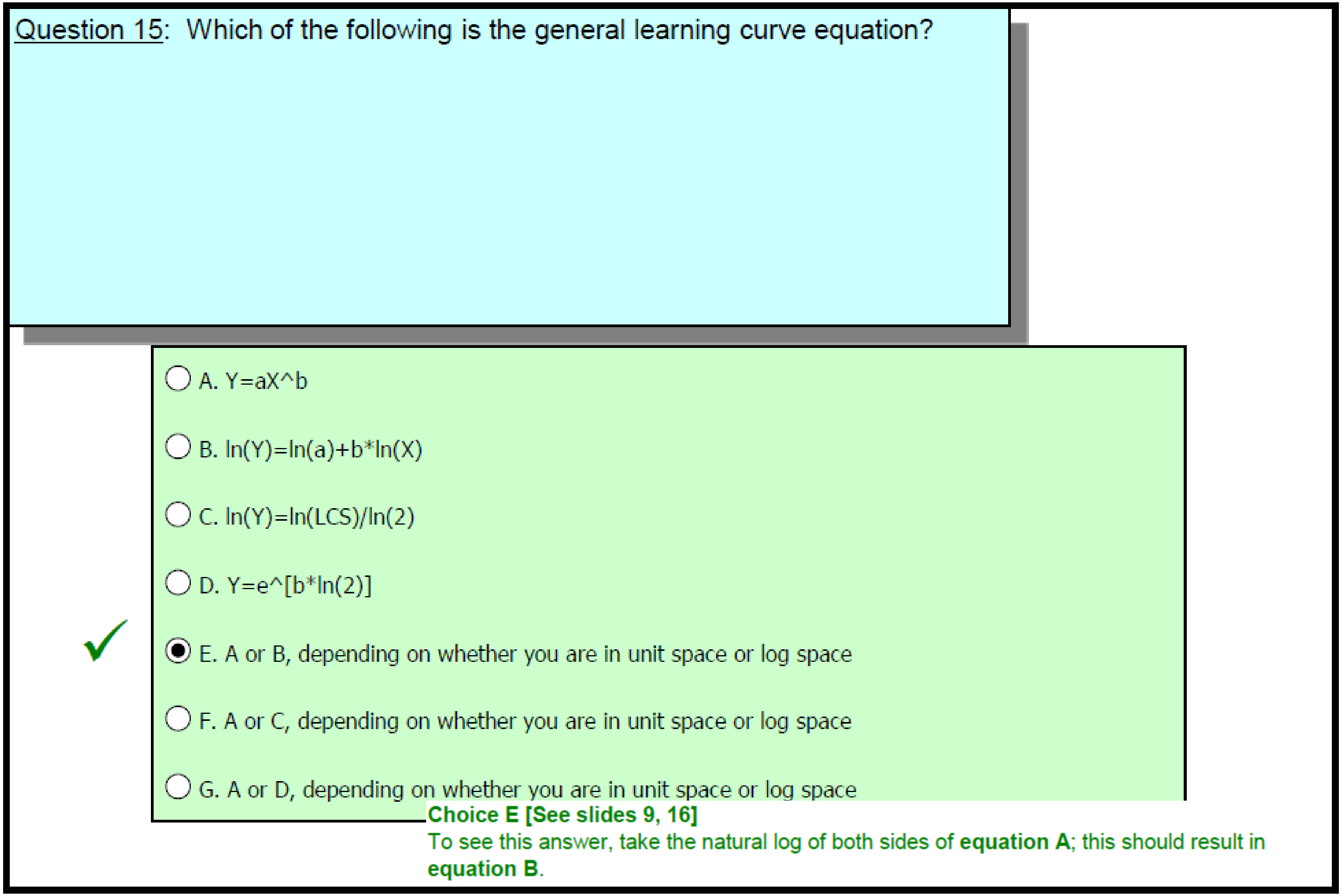
Takeaways:

- **Use mnemonics to remember:**
- **2 Names – Wright & Crawford**
- **CUMAV was first and the ‘Wright brothers’ were first…**
- **ULC must be ‘Crawford’**

Choice C [See slide 31]

Unit learning curve theory was first proposed by James Crawford, hence unit learning curves are also referred to as **Crawford Curves**.

Question 15



Question 15: Which of the following is the general learning curve equation?

○ A. Y=aX^b

○ B. ln(Y)=ln(a)+b*ln(X)

○ C. ln(Y)=ln(LCS)/ln(2)

○ D. Y=e^[b*ln(2)]

✓ ● E. A or B, depending on whether you are in unit space or log space

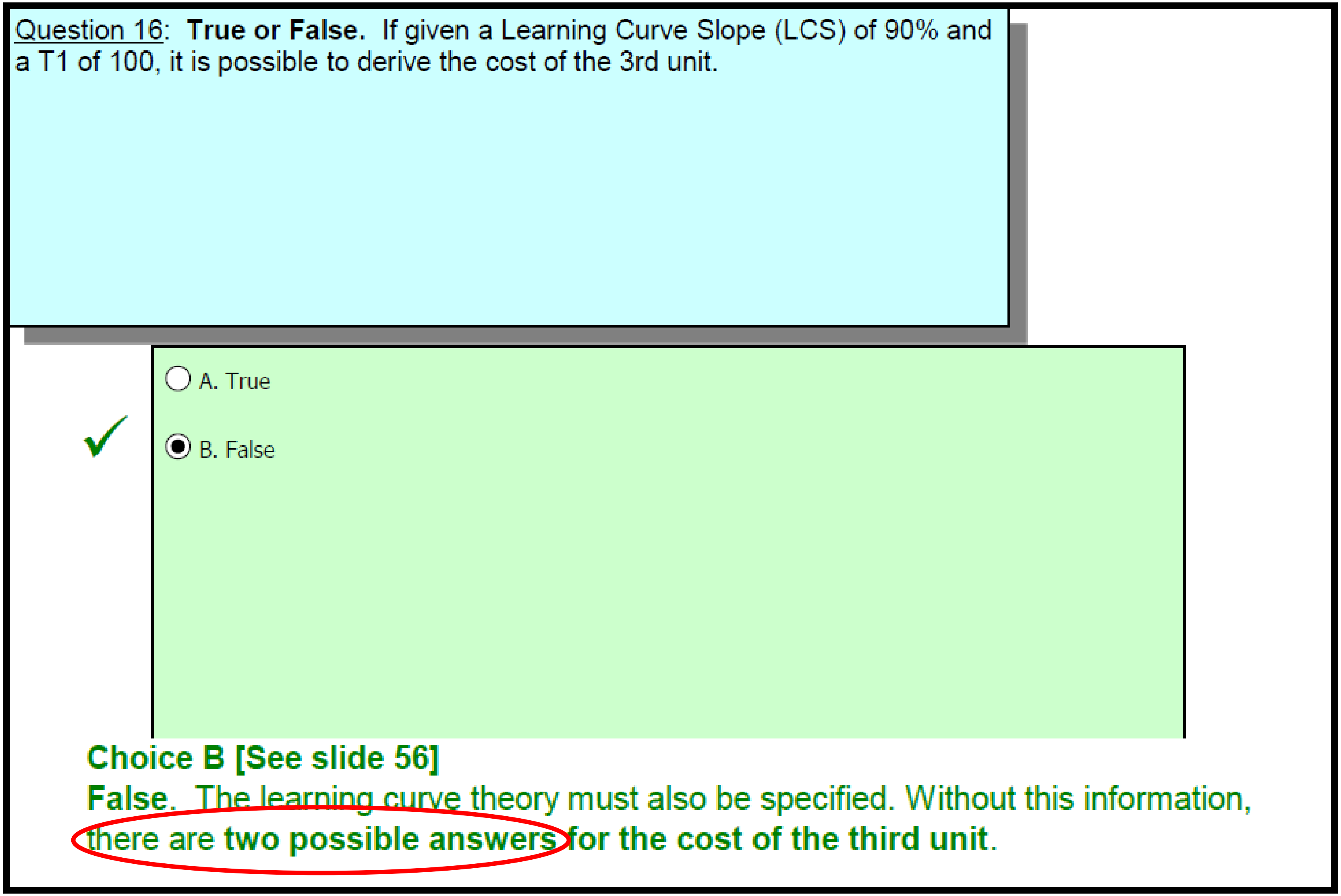
○ F. A or C, depending on whether you are in unit space or log space

○ G. A or D, depending on whether you are in unit space or log space

Choice E [See slides 9, 16]

To see this answer, take the natural log of both sides of **equation A**; this should result in **equation B**.

Question 16



Question 16: **True or False.** If given a Learning Curve Slope (LCS) of 90% and a T1 of 100, it is possible to derive the cost of the 3rd unit.

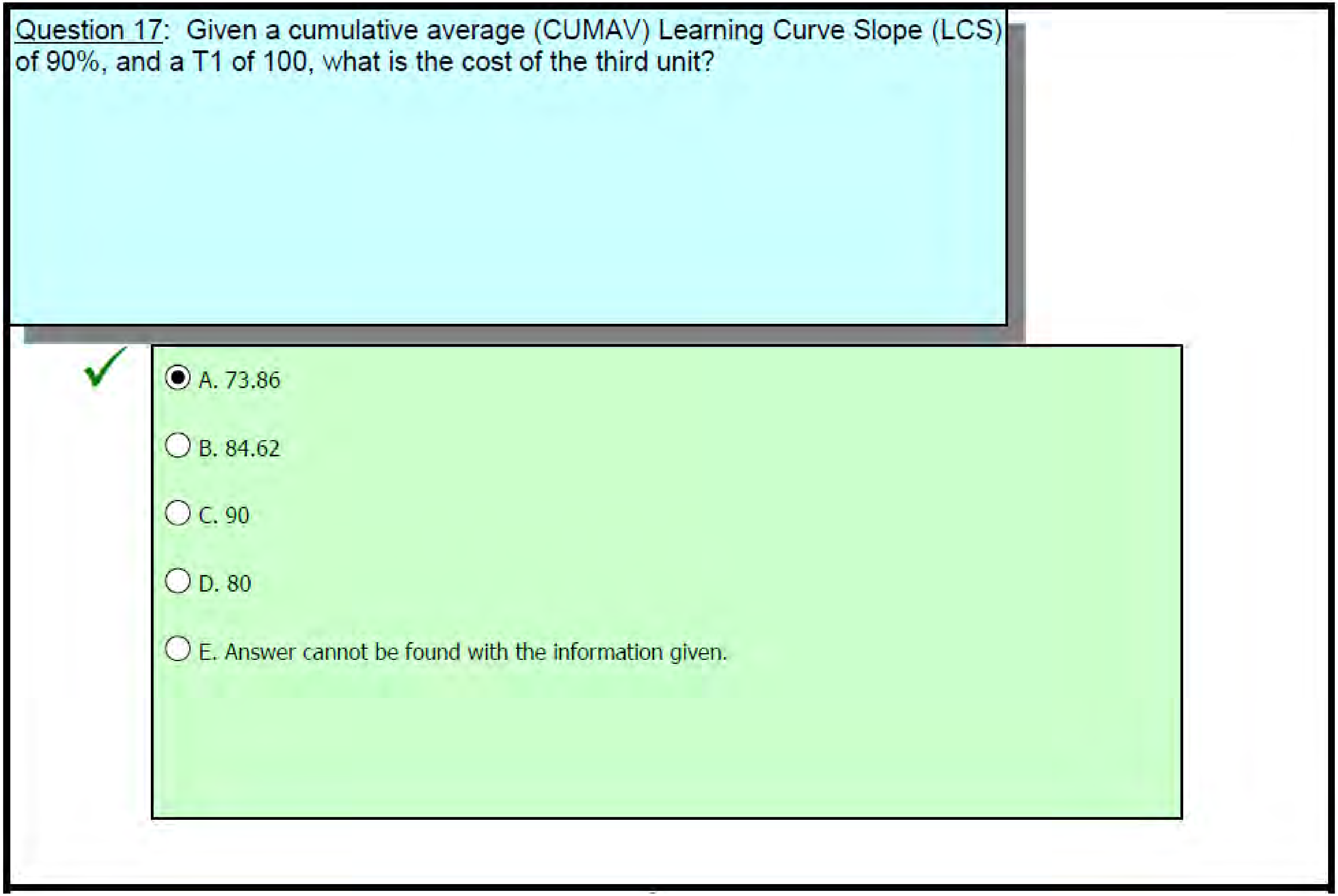
○ A. True

✓ ● B. False

Choice B [See slide 56]

False. The learning curve theory must also be specified. Without this information, there are **two possible answers for the cost of the third unit**.

Question 17



Question 17: Given a cumulative average (CUMAV) Learning Curve Slope (LCS) of 90%, and a T1 of 100, what is the cost of the third unit?

✓

● A. 73.86

○ B. 84.62

○ C. 90

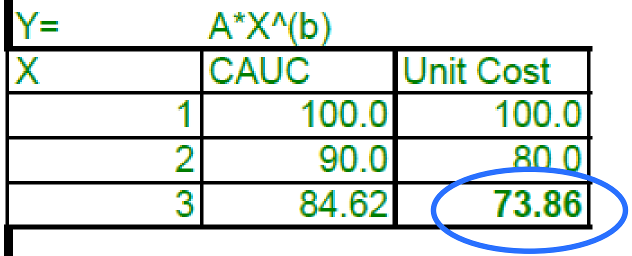
○ D. 80

○ E. Answer cannot be found with the information given.

Question 17

workspace

Choice A [See slide 16]

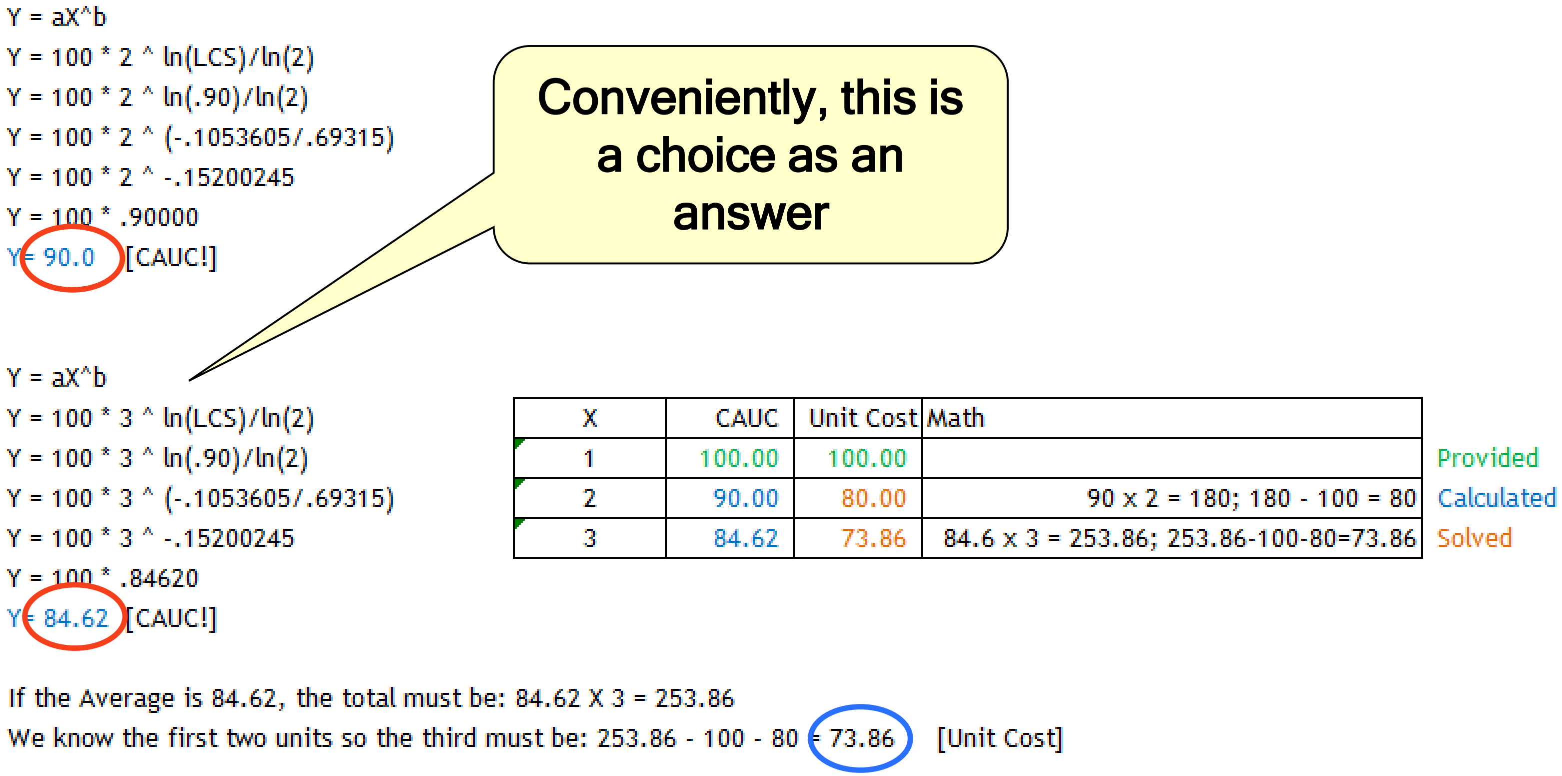


Y= A*X^(b)

X	CAUC	Unit Cost
1	100.0	100.0
2	90.0	80.0
3	84.62	**73.86**

If the cumulative average unit cost of the prior unit is known, and the cumulative average learning curve slope and the T1 are known, the unit cost can be found.

Question 17



Y = aX^b
Y = 100 * 2 ^ ln(LCS)/ln(2)
Y = 100 * 2 ^ ln(.90)/ln(2)
Y = 100 * 2 ^ (-.1053605/.69315)
Y = 100 * 2 ^ -.15200245
Y = 100 * .90000
Y = 90.0 [CAUC!]

Conveniently, this is a choice as an answer

Y = aX^b
Y = 100 * 3 ^ ln(LCS)/ln(2)
Y = 100 * 3 ^ ln(.90)/ln(2)
Y = 100 * 3 ^ (-.1053605/.69315)
Y = 100 * 3 ^ -.15200245
Y = 100 * .84620
Y = 84.62 [CAUC!]

X	CAUC	Unit Cost	Math	
1	100.00	100.00		Provided
2	90.00	80.00	90 x 2 = 180; 180 - 100 = 80	Calculated
3	84.62	73.86	84.6 x 3 = 253.86; 253.86-100-80=73.86	Solved

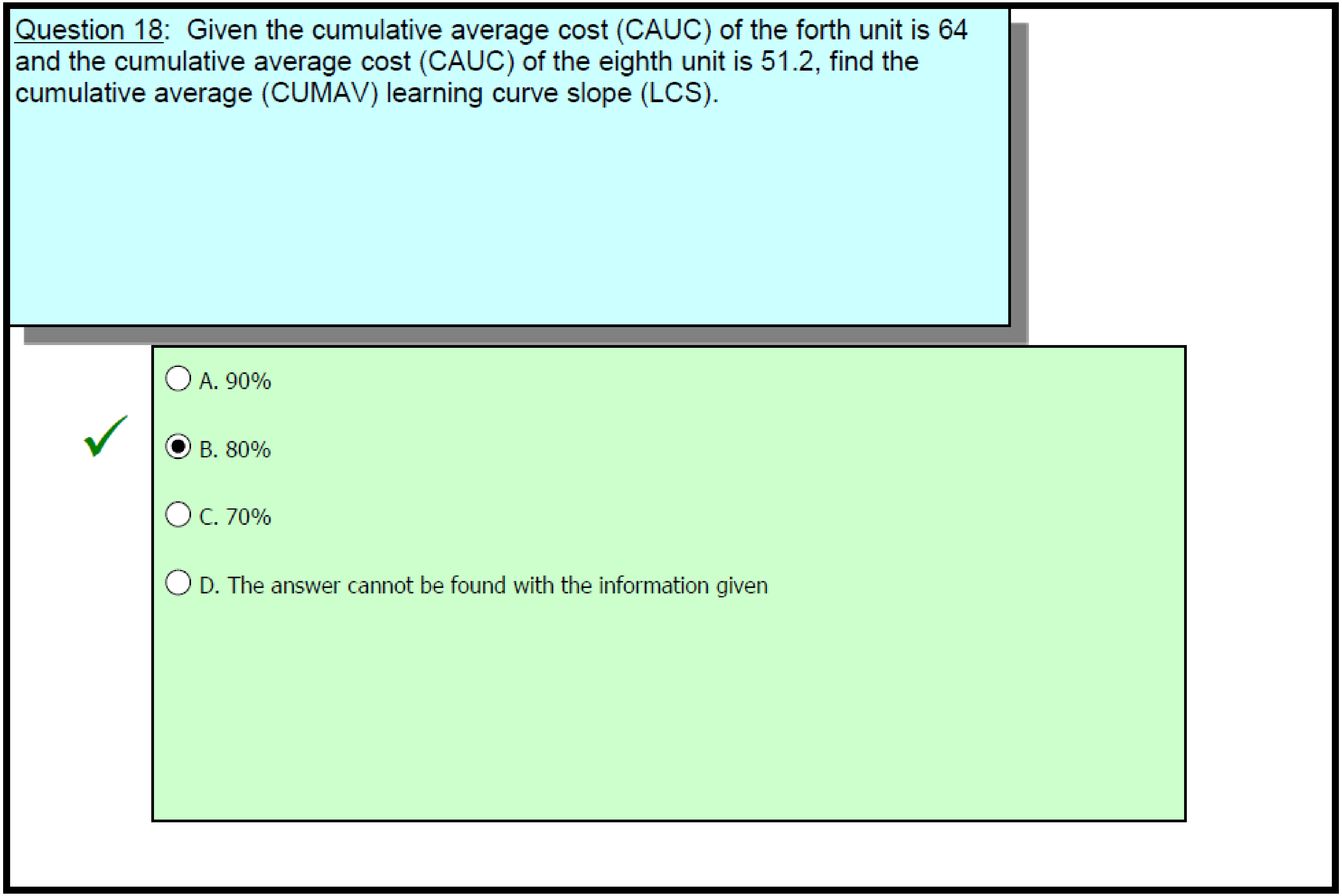
If the Average is 84.62, the total must be: 84.62 X 3 = 253.86

We know the first two units so the third must be: 253.86 - 100 - 80 = 73.86 [Unit Cost]

Takeaways:

- Y= a*X^b
- Use care on CUAC problems – do they want the CUAC or the unit?

Question 18



Question 18: Given the cumulative average cost (CAUC) of the forth unit is 64 and the cumulative average cost (CAUC) of the eighth unit is 51.2, find the cumulative average (CUMAV) learning curve slope (LCS).

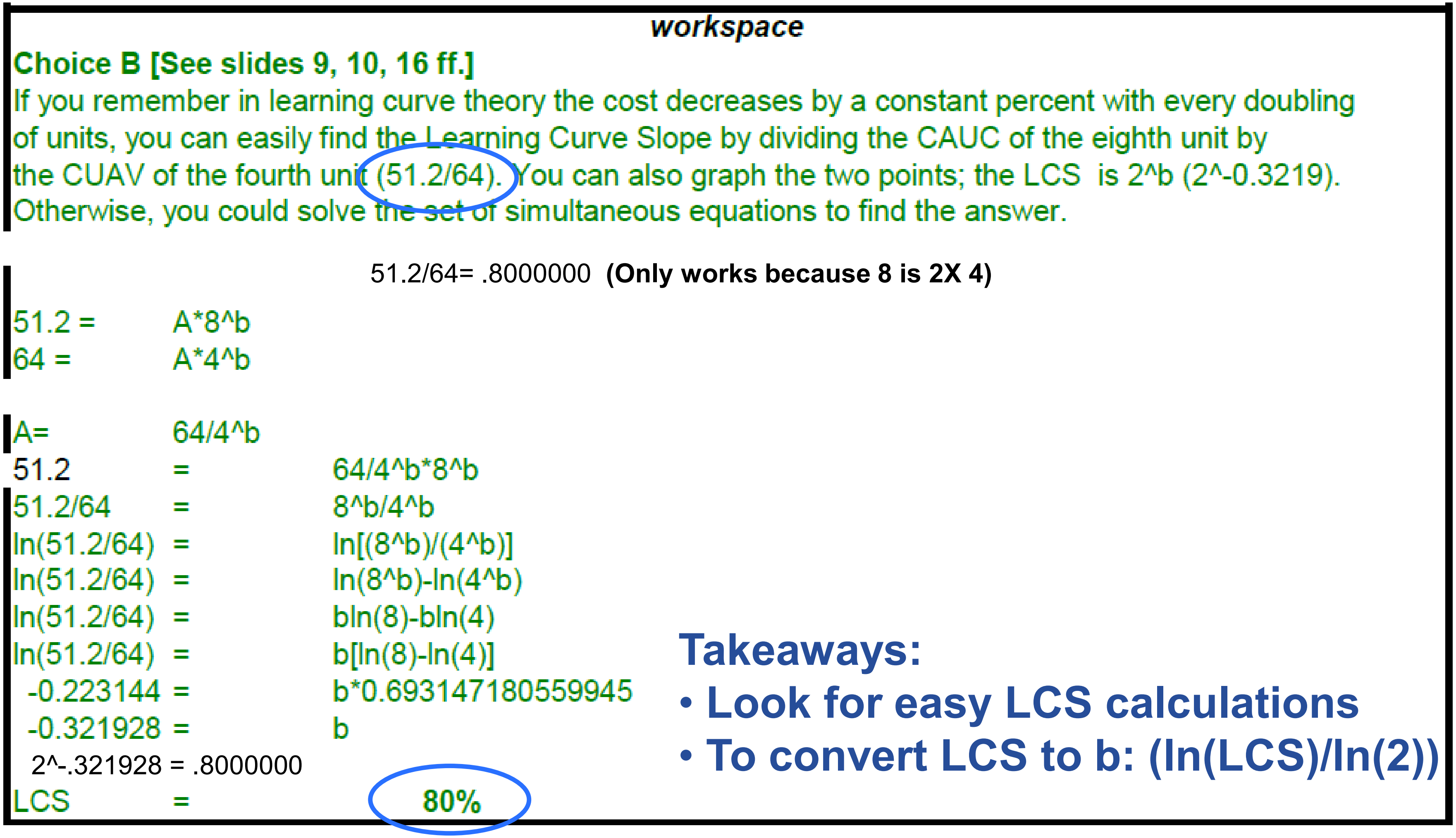
○ A. 90%

✓ ● B. 80%

○ C. 70%

○ D. The answer cannot be found with the information given

Question 18



workspace

Choice B [See slides 9, 10, 16 ff.]

If you remember in learning curve theory the cost decreases by a constant percent with every doubling of units, you can easily find the Learning Curve Slope by dividing the CAUC of the eighth unit by the CUAV of the fourth unit (51.2/64). You can also graph the two points; the LCS is 2^b (2^-0.3219). Otherwise, you could solve the set of simultaneous equations to find the answer.

51.2/64= .8000000 **(Only works because 8 is 2X 4)**

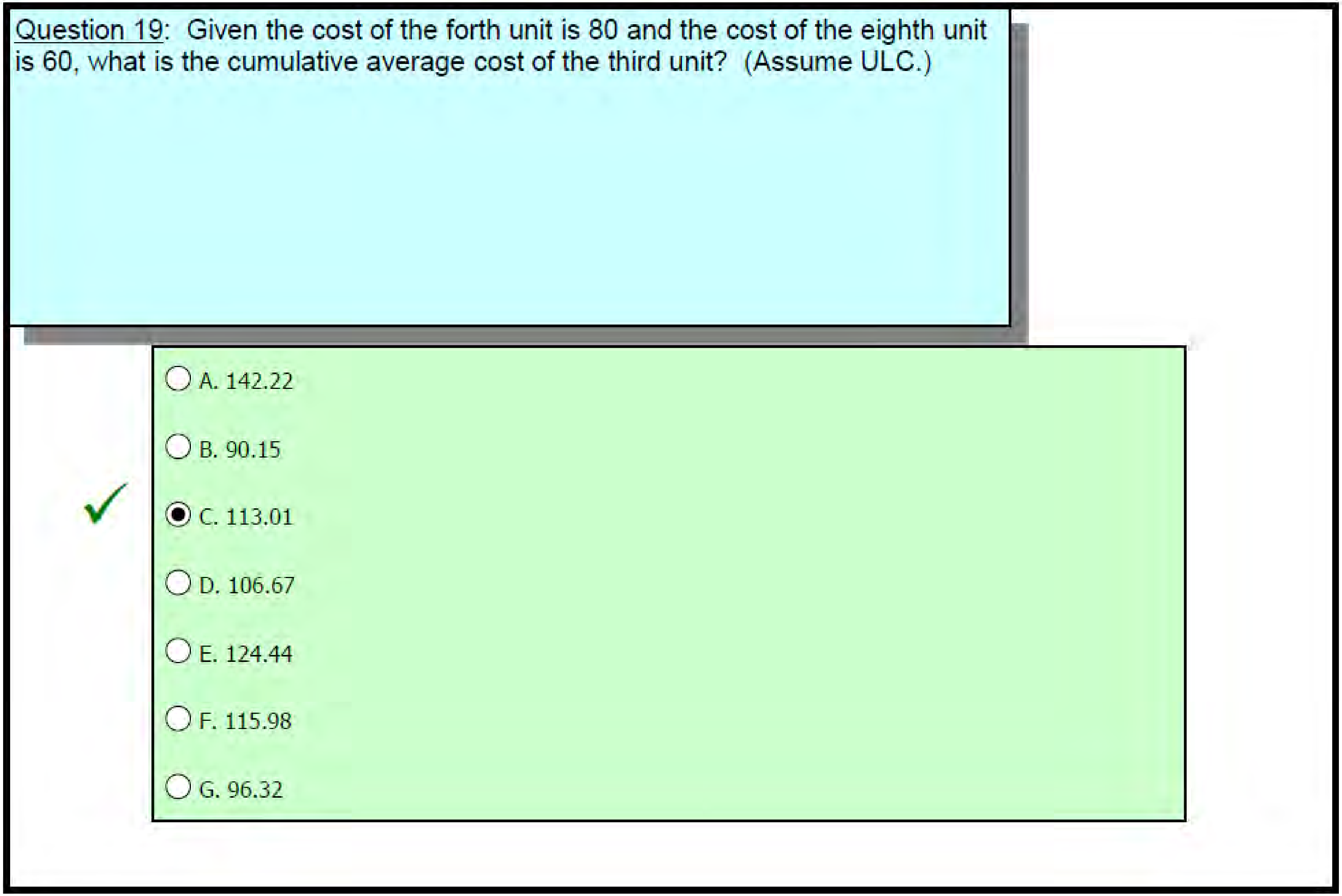
51.2 = A*8^b
64 = A*4^b

A= 64/4^b
51.2 = 64/4^b*8^b
51.2/64 = 8^b/4^b
ln(51.2/64) = ln[(8^b)/(4^b)]
ln(51.2/64) = ln(8^b)-ln(4^b)
ln(51.2/64) = bln(8)-bln(4)
ln(51.2/64) = b[ln(8)-ln(4)]
-0.223144 = b*0.693147180559945
-0.321928 = b
2^-.321928 = .8000000
LCS = **80%**

Takeaways:

- **Look for easy LCS calculations**
- **To convert LCS to b: (ln(LCS)/ln(2))**

Question 19



Question 19: Given the cost of the forth unit is 80 and the cost of the eighth unit is 60, what is the cumulative average cost of the third unit? (Assume ULC.)

○ A. 142.22

○ B. 90.15

✓ ● C. 113.01

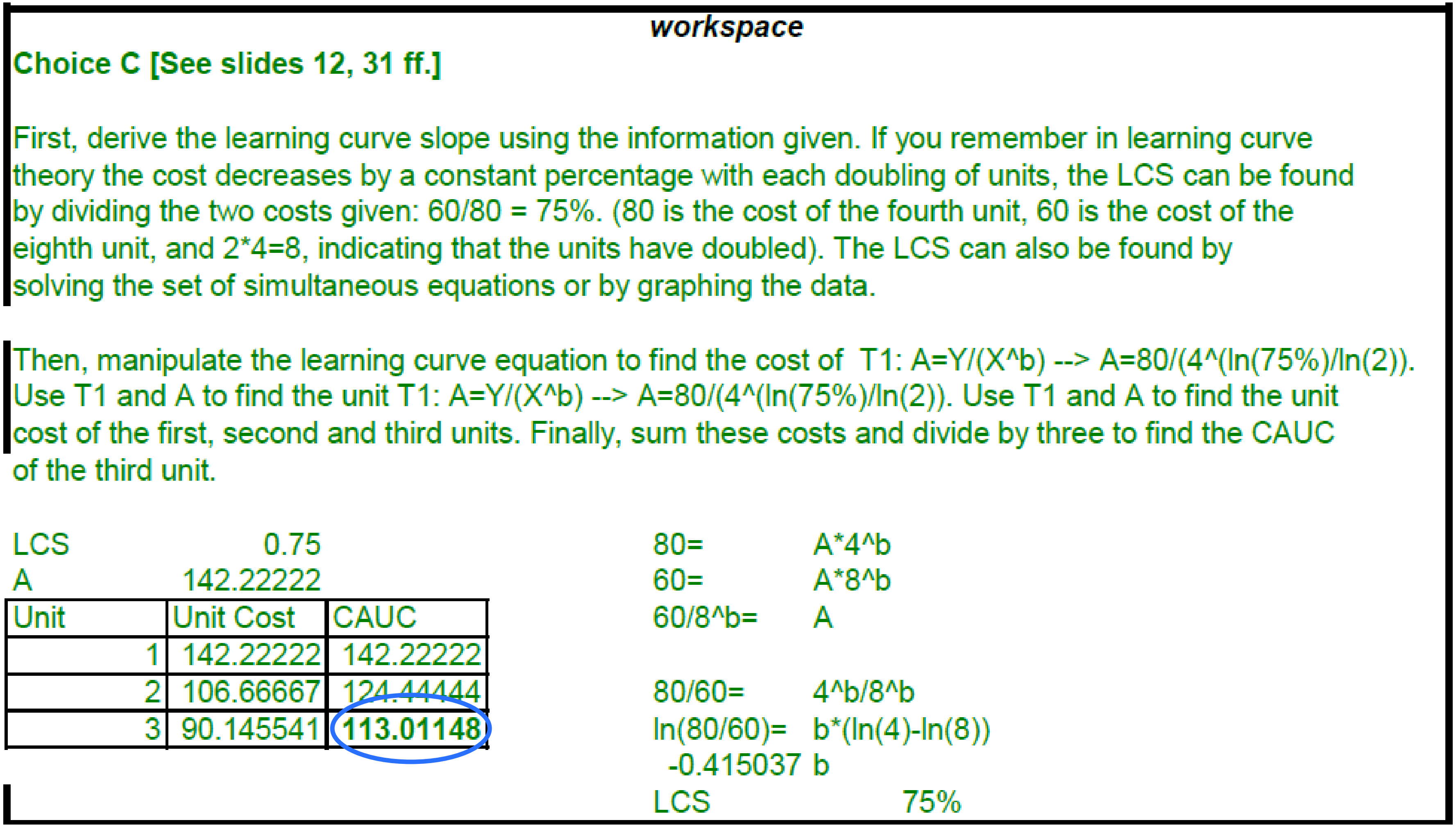
○ D. 106.67

○ E. 124.44

○ F. 115.98

○ G. 96.32

Question 19



workspace

Choice C [See slides 12, 31 ff.]

First, derive the learning curve slope using the information given. If you remember in learning curve theory the cost decreases by a constant percentage with each doubling of units, the LCS can be found by dividing the two costs given: 60/80 = 75%. (80 is the cost of the fourth unit, 60 is the cost of the eighth unit, and 2*4=8, indicating that the units have doubled). The LCS can also be found by solving the set of simultaneous equations or by graphing the data.

Then, manipulate the learning curve equation to find the cost of T1: A=Y/(X^b) --> A=80/(4^(ln(75%)/ln(2)). Use T1 and A to find the unit T1: A=Y/(X^b) --> A=80/(4^(ln(75%)/ln(2)). Use T1 and A to find the unit cost of the first, second and third units. Finally, sum these costs and divide by three to find the CAUC of the third unit.

LCS 0.75
A 142.22222

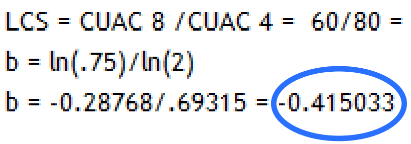
Unit	Unit Cost	CAUC
1	142.22222	142.22222
2	106.66667	124.44444
3	90.145541	**113.01148**

80= A*4^b
60= A*8^b
60/8^b= A

80/60= 4^b/8^b
ln(80/60)= b*(ln(4)-ln(8))
-0.415037 b
LCS 75%

Question 19

Solve for b

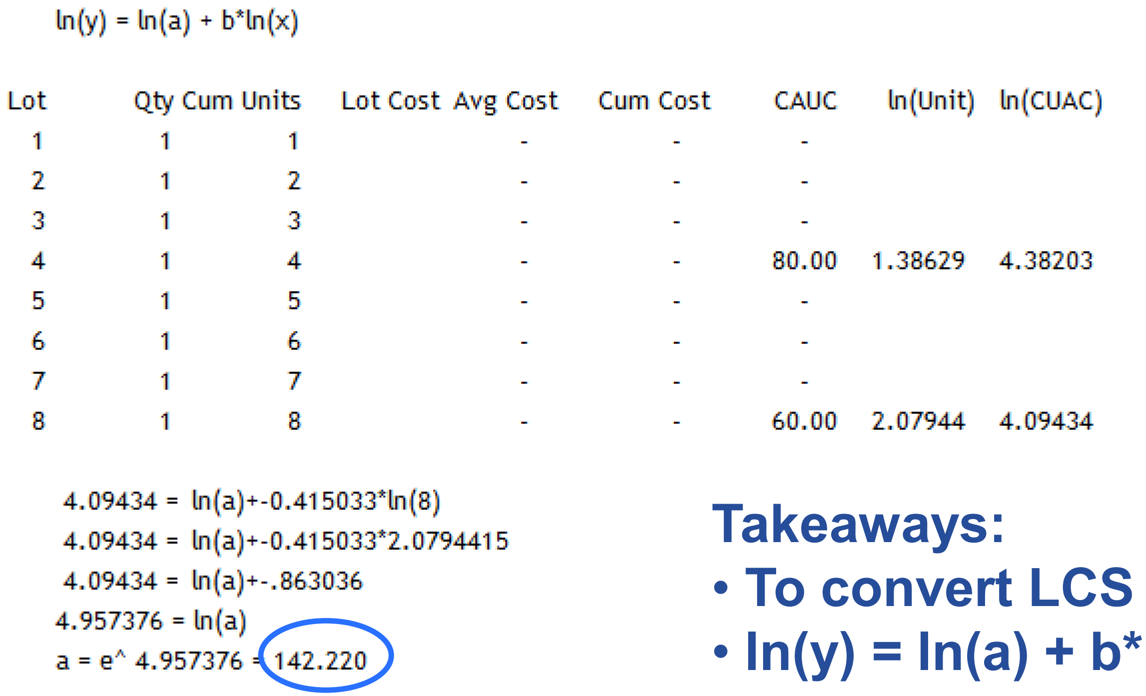


LCS = CUAC 8 /CUAC 4 = 60/80 = .75

b = ln(.75)/ln(2)

b = -0.28768/.69315 = -0.415033

Solve for a



ln(y) = ln(a) + b*ln(x)

Lot	Qty	Cum Units	Lot Cost	Avg Cost	Cum Cost	CAUC	ln(Unit)	ln(CUAC)
1	1	1		-	-	-		
2	1	2		-	-	-		
3	1	3		-	-	-		
4	1	4		-	-	80.00	1.38629	4.38203
5	1	5		-	-	-		
6	1	6		-	-	-		
7	1	7		-	-	-		
8	1	8		-	-	60.00	2.07944	4.09434

4.09434 = ln(a)+-0.415033*ln(8)

4.09434 = ln(a)+-0.415033*2.0794415

4.09434 = ln(a)+-.863036

4.957376 = ln(a)

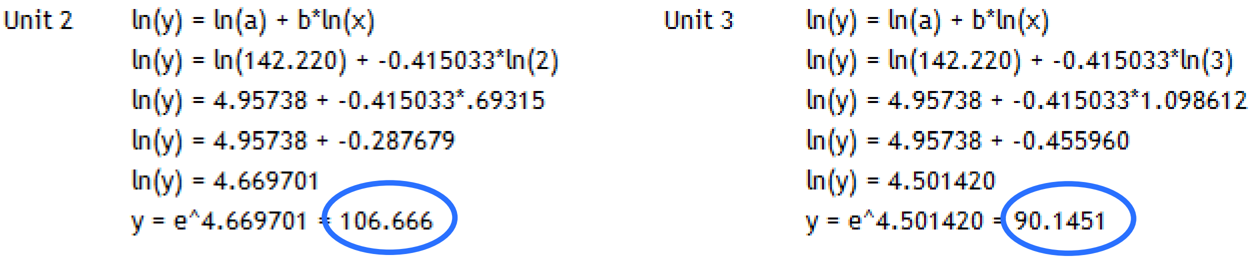
a = e^ 4.957376 = 142.220

Takeaways:

- **To convert LCS to b: (ln(LCS)/ln(2))**
- **ln(y) = ln(a) + b*ln(x)**

Question 19

Solve for units 2 and 3



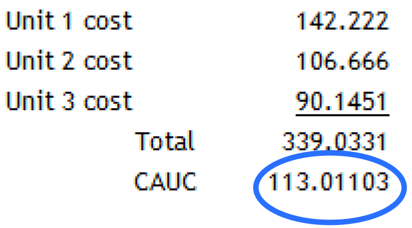
Unit 2

ln(y) = ln(a) + b*ln(x)
ln(y) = ln(142.220) + -0.415033*ln(2)
ln(y) = 4.95738 + -0.415033*.69315
ln(y) = 4.95738 + -0.287679
ln(y) = 4.669701
y = e^4.669701 = 106.666

Unit 3

ln(y) = ln(a) + b*ln(x)
ln(y) = ln(142.220) + -0.415033*ln(3)
ln(y) = 4.95738 + -0.415033*1.098612
ln(y) = 4.95738 + -0.455960
ln(y) = 4.501420
y = e^4.501420 = 90.1451

Solve for CAUC at 3

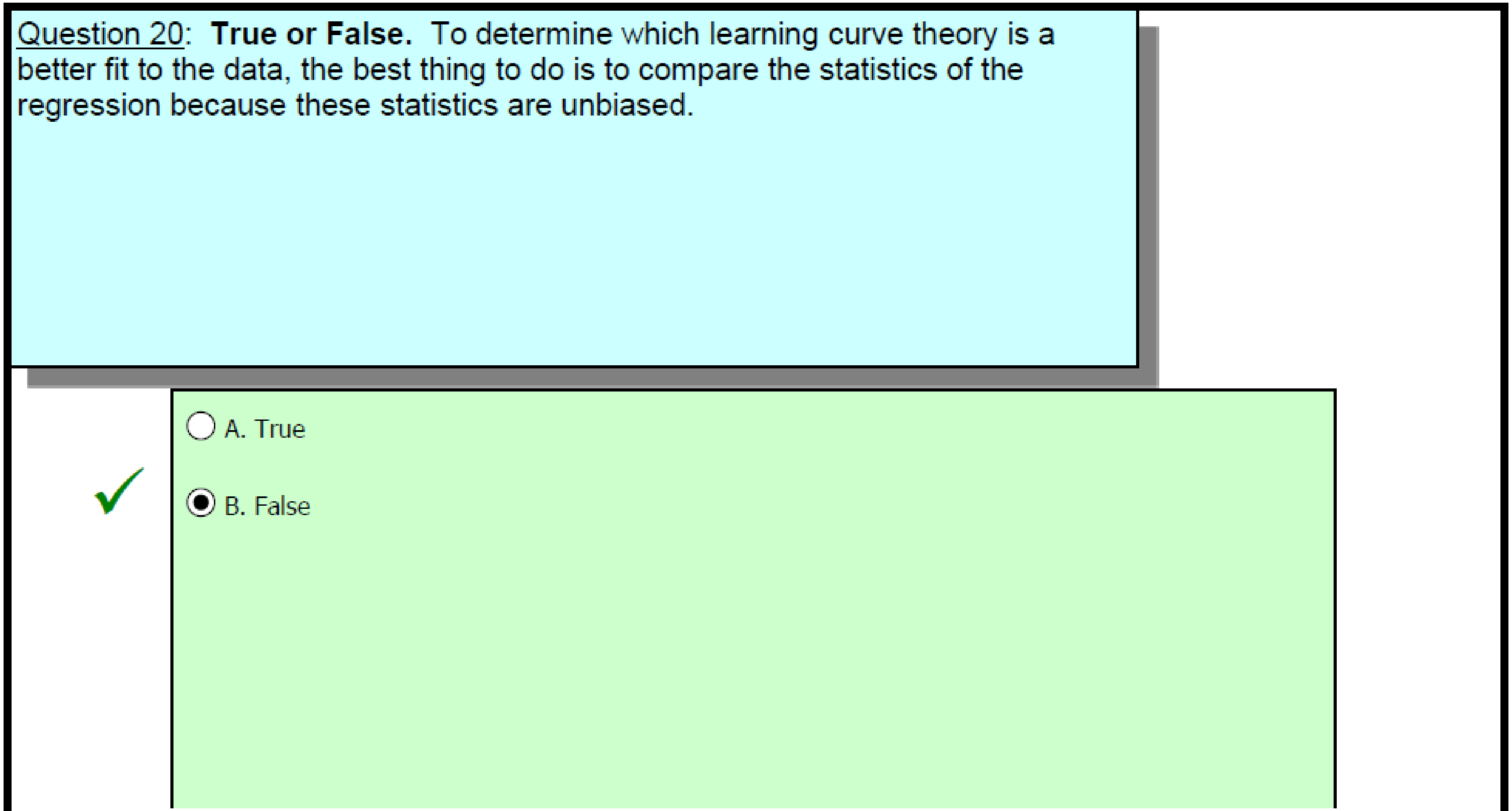


Unit 1 cost		142.222
Unit 2 cost		106.666
Unit 3 cost		90.1451
	Total	339.0331
	CAUC	113.01103

Takeaways:

- **ln(y) = ln(a) + b*ln(x)**
- **To convert ln(y) to y raise e^ln(y)**
- **Use care on CUAC problems – do they want the CUAC or the unit?**

Question 20



Question 20: **True or False.** To determine which learning curve theory is a better fit to the data, the best thing to do is to compare the statistics of the regression because these statistics are unbiased.

○ A. True

✓ ● B. False

Choice B [See slide 14 ff.]

False. Though the statistics of the regression are used to determine which theory is a better fit, it is importantto keep in mind that the **statistics of the regression inherently favor the cumulative average learning curve theory**.

Related and Advanced Topics

- Laws of Logarithms
- Learning Curve Data
- Lot Midpoint
  - Rules of Thumb
  - Formula Derivation
- Production Break Effects
- New Work Effects
- Production Rate Effects
- “Meta Learning Curves”

Laws of Logarithms

"Log is inverse of exponential"

$$\log_b\left(b^x\right)=b^{\log_b x}=x$$

"Log of a product"

$$\log_b(xy)=\log_b x+\log_b y$$

"Log of a quotient"

$$\log_b\left(\frac{x}{y}\right)=\log_b x-\log_b y$$

"Log of a power"

$$\log_b\left(x^y\right)=y\log_b x$$

"Change of base"

$$\log_b a=\frac{\log_c a}{\log_c b} \qquad \log_x y=\frac{1}{\log_y x}$$

Learning Curve Concepts – Data Conversion

- You can calculate just about any learning cost data type from any other
  - Cannot recover unit data from lot data

Tip: Do not memorize these formulae!

	UC	Cum	CAUC	AUC	Lot
UC_n	UC_n	$Cum_n - Cum_{n-1}$	$nCAUC_n - (n-1)CAUC_{n-1}$	??	??
Cum_n	$\sum_{i=1}^{n} UC_i$	Cum_n	$nCAUC_n$	$Cum_{F-1} + (L-F+1)AUC_{F,L}$	$Cum_{F-1} + Lot_{F,L}$
$CAUC_n$	$\sum_{i=1}^{n} UC_i \Big/ n$	$Cum_n \big/ n$	$CAUC_n$	$\left[\begin{matrix}(F-1)CAUC_{F-1} + \\ (L-F+1)AUC_{F,L}\end{matrix}\right] \Big/ L$	$\left[\begin{matrix}(F-1)CAUC_{F-1} + \\ Lot_{F,L}\end{matrix}\right] \Big/ L$
$AUC_{F,L}$	$\sum_{i=F}^{L} UC_i \Big/ (L-F+1)$	$\frac{Cum_L - Cum_{F-1}}{(L-F+1)}$	$\frac{\left[\begin{matrix}L \cdot CAUC_L - \\ (F-1)CAUC_{F-1}\end{matrix}\right]}{(L-F+1)}$	$AUC_{F,L}$	$\frac{Lot_{F,L}}{(L-F+1)}$
$Lot_{F,L}$	$\sum_{i=F}^{L} UC_i$	$Cum_L - Cum_{F-1}$	$L \cdot CAUC_L - (F-1)CAUC_{F-1}$	$(L-F+1) \cdot AUC_{F,L}$	$Lot_{F,L}$

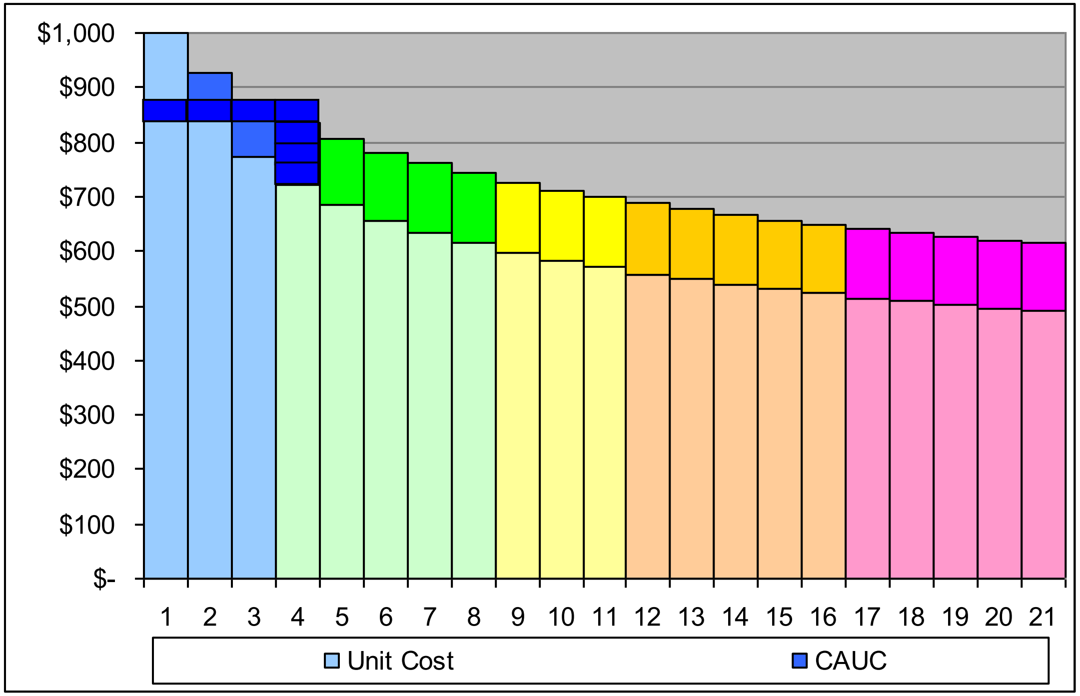
v1.2

Learning Curve Concepts – Unit and CUMAV Data Illustration

- Note that CUMAV Cost (CAUC) is always a more gradual curve than Unit Cost

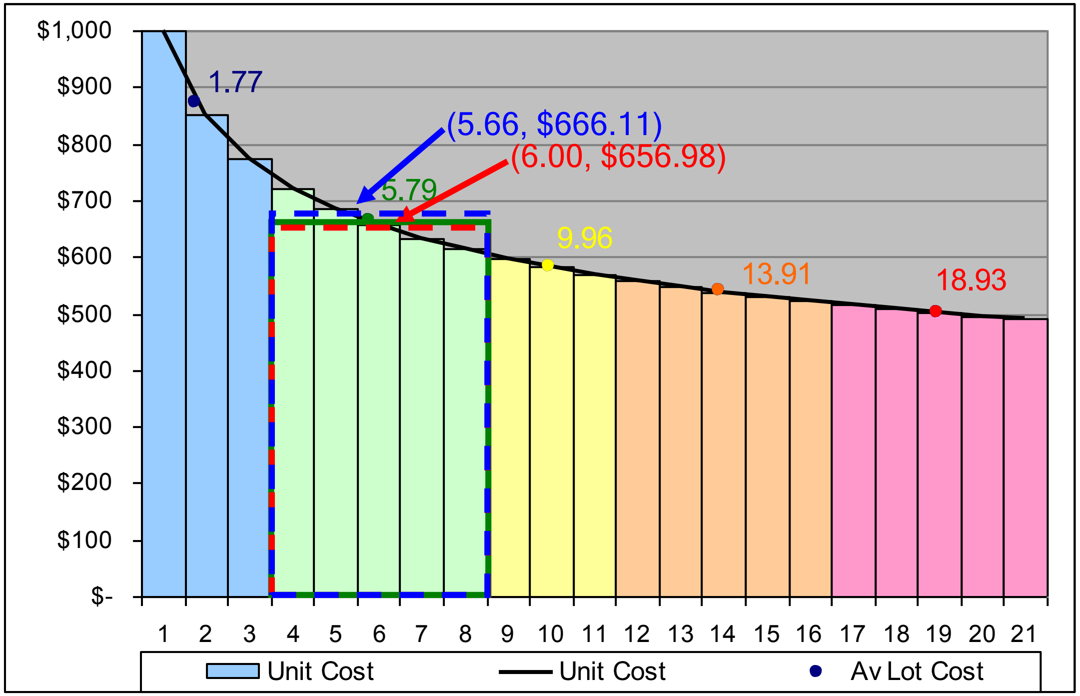
$$CAUC_n = \frac{(n-1)CAUC_{n-1} + UC_n}{n} = CAUC_{n-1} - \frac{CAUC_{n-1} - UC_n}{n}$$

Unit	Unit Cost	CAUC
1	$ 1,000.00	$ 1,000.00
2	$ 850.00	$ 925.00
3	$ 772.91	$ 874.30
4	$ 722.50	$ 836.35
5	$ 685.67	$ 806.22
6	$ 656.98	$ 781.34
7	$ 633.66	$ 760.25
8	$ 614.13	$ 741.98
9	$ 597.40	$ 725.92
10	$ 582.82	$ 711.61
11	$ 569.94	$ 698.73
12	$ 558.43	$ 687.04
13	$ 548.05	$ 676.34
14	$ 538.61	$ 666.51
15	$ 529.97	$ 657.40
16	$ 522.01	$ 648.94
17	$ 514.64	$ 641.04
18	$ 507.79	$ 633.64
19	$ 501.39	$ 626.68
20	$ 495.40	$ 620.11
21	$ 489.76	$ 613.91



Lot Midpoint Comparison

GM	Cost (GM)	LMP	Cost (LMP)	AM	Cost (AM)
1.73	$ 879.16	1.77	$ 874.30	2.00	$ 850.00
1.73	$ 879.16	1.77	$ 874.30	2.00	$ 850.00
1.73	$ 879.16	1.77	$ 874.30	2.00	$ 850.00
5.66	$ 666.11	5.79	$ 662.59	6.00	$ 656.98
5.66	$ 666.11	5.79	$ 662.59	6.00	$ 656.98
5.66	$ 666.11	5.79	$ 662.59	6.00	$ 656.98
5.66	$ 666.11	5.79	$ 662.59	6.00	$ 656.98
5.66	$ 666.11	5.79	$ 662.59	6.00	$ 656.98
9.95	$ 583.51	9.96	$ 583.39	10.00	$ 582.82
9.95	$ 583.51	9.96	$ 583.39	10.00	$ 582.82
9.95	$ 583.51	9.96	$ 583.39	10.00	$ 582.82
13.86	$ 539.91	13.91	$ 539.41	14.00	$ 538.61
13.86	$ 539.91	13.91	$ 539.41	14.00	$ 538.61
13.86	$ 539.91	13.91	$ 539.41	14.00	$ 538.61
13.86	$ 539.91	13.91	$ 539.41	14.00	$ 538.61
13.86	$ 539.91	13.91	$ 539.41	14.00	$ 538.61
18.89	$ 502.05	18.93	$ 501.80	19.00	$ 501.39
18.89	$ 502.05	18.93	$ 501.80	19.00	$ 501.39
18.89	$ 502.05	18.93	$ 501.80	19.00	$ 501.39
18.89	$ 502.05	18.93	$ 501.80	19.00	$ 501.39
18.89	$ 502.05	18.93	$ 501.80	19.00	$ 501.39



The average (arithmetic mean) overestimates LMP, whereas the geometric mean underestimates LMP

Lot Midpoint Formula Derivation

- Problem: The average unit cost for a lot, upon which the LMP calculation relies, is an open-ended sum
- Solution: Use the area under the curve (definite integral) to approximate the sum, enabling a closed-form formula!

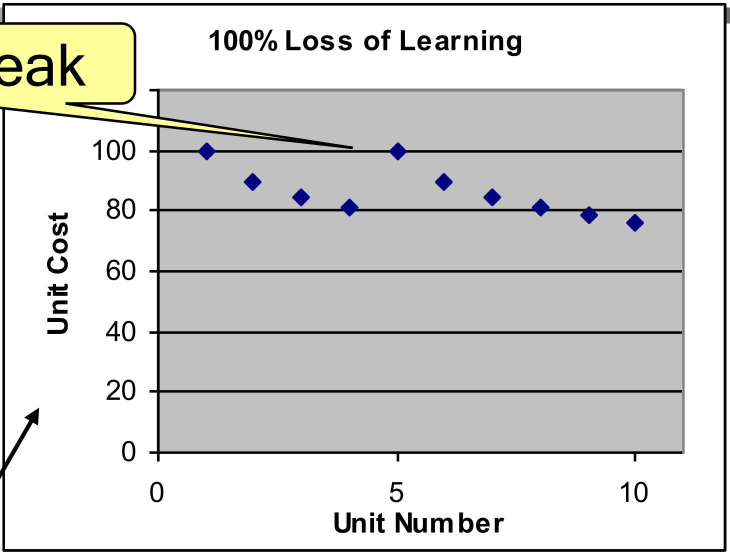
$$\sum_{i=F}^{L} Ai^{b} \approx A\int_{F-\frac{1}{2}}^{L+\frac{1}{2}} x^{b}\,dx = A\frac{x^{b+1}}{b+1}\Bigg|_{F-\frac{1}{2}}^{L+\frac{1}{2}} = A\frac{\left(L+\frac{1}{2}\right)^{b+1}-\left(F-\frac{1}{2}\right)^{b+1}}{b+1}$$

$$LMP = \left(\frac{\sum_{i=F}^{L} i^{b}}{N}\right)^{\left(\frac{1}{b}\right)} \approx \left[\frac{\left(L+\frac{1}{2}\right)^{b+1}-\left(F-\frac{1}{2}\right)^{b+1}}{N(b+1)}\right]^{\left(\frac{1}{b}\right)}$$

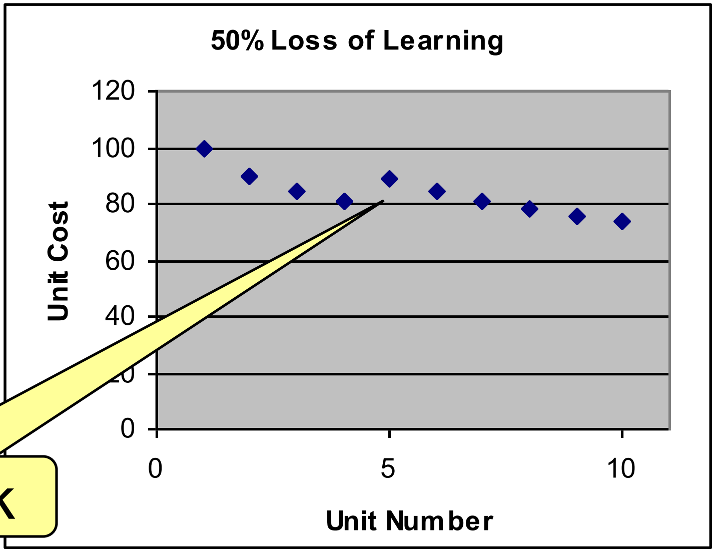
Effects of Production Breaks

11

- A production break occurs when the manufacturer shuts down the production line for a period of time
- The possible effects of production breaks are:
  - No change
  - Complete loss of learning
  - Partial loss of learning



100% Break



50% Break

Break Analysis

- There are two equivalent conceptual approaches to addressing break analysis
  - The first is to view loss of learning as a percent loss (focus on y-axis = cost)
  - The second is to view loss of learning as a reversion to a previous unit number (focus on x-axis = unit number)
- Next we will illustrate loss of learning using the first approach and this information:
  - T1 = 100, LCS = 90% (b = -0.152)
  - Break occurs at unit 5
  - 50% loss of learning
  - Unit Theory

Break Analysis

- When a break occurs, you must first identify your break unit cost (i.e., the cost of the first unit after the break):

$$a' = aN^b + L(a - aN^b) = La + (1-L)aN^b$$

a’ = Break unit cost

a = Beginning T1 Value

L = % of Learning Lost

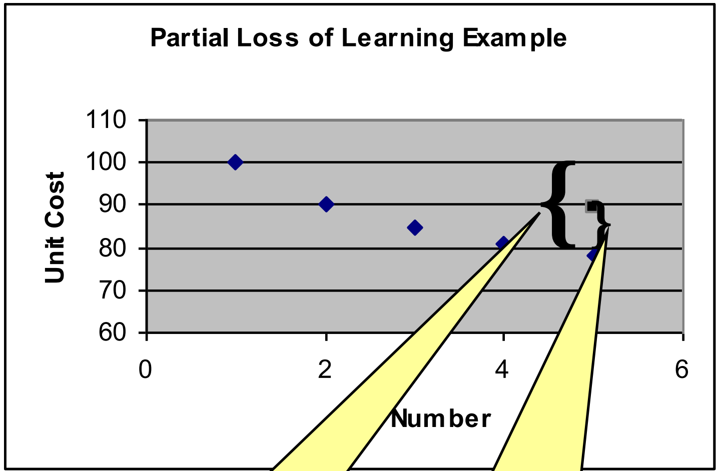
N = First unit after break

b = learning curve exponent

- For our example:

  $a' = 0.5*100 + (1-0.5)*100*5^{-0.152}$

  $a' = 89.15$



Total learning at unit 5

50% of learning through unit 5

Break Analysis

- Once you have the a’, you next must solve for your effective unit number Q

$$a' = aQ^{b} \qquad Q = (a'/a)^{(1/b)}$$

a’ = cost of unit following break

a = old T1

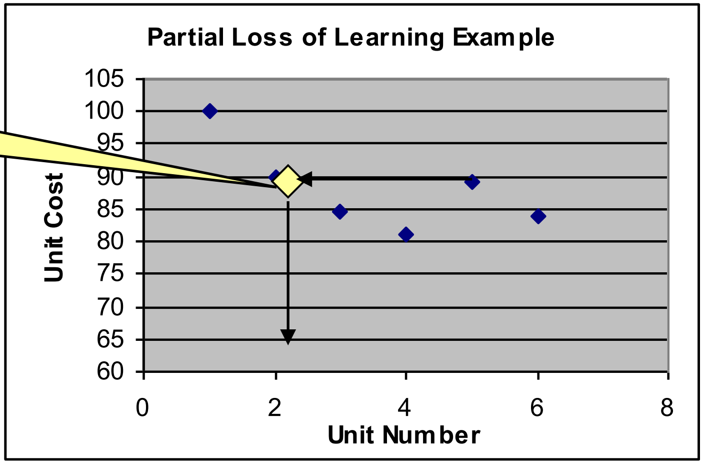
Q = effective new unit number

b = learning curve exponent

  - Continuing our example:

    $Q = (89.15/100)^{(1/-0.152)} = 2.128$

Point Q



- Now treat future units as Q+(X-N), so the unit cost of the unit following the break unit would be:

$Y = 100 * (2.128 + (6 - 5))^{-0.152} = 84.081$

Break Analysis – Anderlohr

- Expert-based Production Break methodology due to George Anderlohr
- Quantifies degree of loss of learning in each of five areas:
  1. Personnel Learning
  2. Supervisory Learning
  3. Continuity of Productivity
  4. Methods
  5. Special Tooling
- Total percent learning lost is a weighted average
- “Retrograde Method” applied to determine equivalent unit number

Anderlohr, G. 1969. "What Production Breaks Cost," Industrial Engineering 1(9):34-36.
Anderlohr, G. (1969), "Determining the cost of production breaks", Management Review, Vol. 58 No.12, pp.16-9.

> “Having said my piece I suggest you go into the sunshine, find something to laugh about and live a happy life.”
> -George Anthony Anderlohr, 31 Jan 1917 - 24 Feb 2008

http://www.georgeanderlohr.com

Effects of Adding New Work

- Another important impact on learning is the addition of new work
- One method of adjusting for new work is to run an additional curve for the new work, starting at the affected unit, and add it to the old curve

$$Y = a_1 X^{b_1} + a_2 (X - L)^{b_2}$$

Y = unit cost

a_1 = original T1 value

a_2 = new work T1 value

X = current unit number

L = last unit before addition of new work

b_1 = exponent for original LCS

b_2 = exponent for new work LCS (typically the same LCS is used)