Multiplicative-Form CERs: A Progress Review

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Project Motivations and Goals

- The exponential function, $y = a \times x^b$, and the power function, $y = a \times b^{cx}$, are increasingly popular model forms for CER development
- The Excel Solver has put the ease of fitting these model forms on a par with linear regressions
  - Little training is necessary to "push the [Solve] button"
- However, the math of "multiplicative CERs" is less easy and knowledge about them is less widespread than their linear regression cousins
  - Compare courses you've taken; books on your shelf
- Now is a good time to bring together key facts and lessons about working with these functions

Today's Work

- **Report findings on model fitting exercises**
  - **Defer "theoretical" work for later**
  - **"Practical" exercises often surprise**
- **Keep the initial focuses simple and practical**
  - **Fit model parameters – "constants" – with the Excel Solver**
  - **Vary:**
    - **Model constants *a* and *b* in an artificial data set – let *c* = 1**
    - **Initial trial values the Solver uses for the model constants**
    - **Objective functions – the criteria for "best" model constants**
    - **Solver option settings**
- **Observe useful and interesting results**
  - **How do the fitted constants relate to the constants in the data set?**
  - **How do the initial model constants affect the results?**
  - **How do different Solver option settings affect the results?**
  - **How do different objective functions affect the results?**

Exercise Procedures

Exercise Procedures

- **Design a data set of *a*s, *b*s, and *x*s**
- **Spreadsheet formulas populate the *y*s**

Data area for $y = ax^b$

a	x	b	$y=ax^b$
1.50	2.00	-0.74	0.90
1.50	4.00	-0.74	0.54
1.50	2.00	-0.15	1.35
1.50	4.00	-0.15	1.22
2.50	2.00	-0.74	1.50
2.50	4.00	-0.74	0.90
2.50	2.00	-0.15	2.25
2.50	4.00	-0.15	2.03

Exercise Procedures

- **Provide starting values *a'*,*b'* for *a*,*b***
- **Spreadsheet formulas populate *y'* and *error functions***

Working area

a'	1.00000	x	$y' = a' x^{b'}$	y	$(y-y') / y'$	$((y-y') / y')^2$
b'	0.00000	2.00	1.00	0.90	-0.10	0.01
		4.00	1.00	0.54	-0.46	0.21
		2.00	1.00	1.35	0.35	0.12
		4.00	1.00	1.22	0.22	0.05
		2.00	1.00	1.50	0.50	0.25
		4.00	1.00	0.90	-0.10	0.01
		2.00	1.00	2.25	1.25	1.56
		4.00	1.00	2.03	1.03	1.05

Exercise Procedures

- **Formulas populate the objective and other error summaries**

Working area

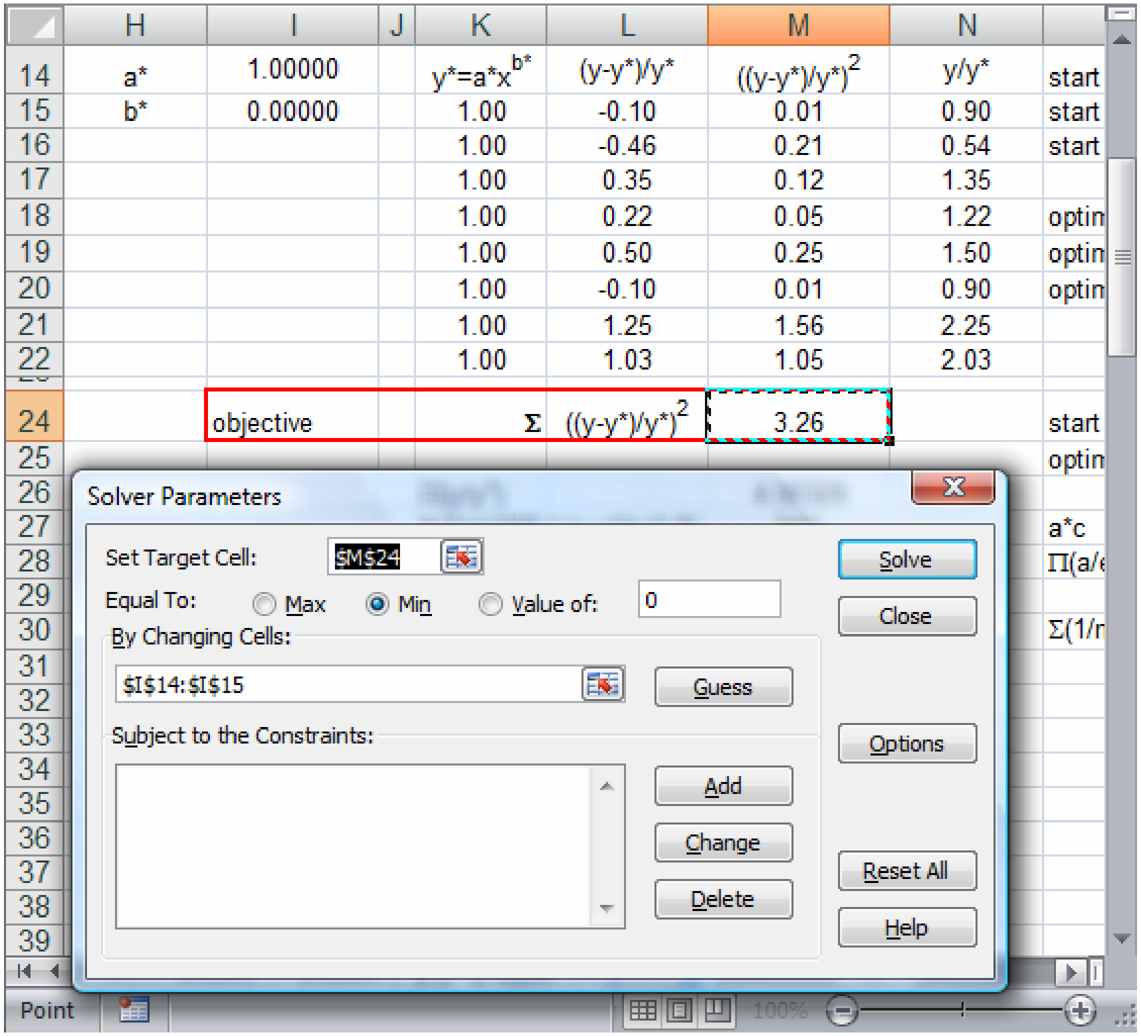
a'	1.00000	x	$y' = a' x^{b'}$	y	$(y-y') / y'$	$((y-y') / y')^2$
b'	0.00000	2.00	1.00	0.9	-0.10	0.01
		4.00	1.00	0.54	-0.46	0.21
		2.00	1.00	1.35	0.35	0.12
		4.00	1.00	1.215	0.22	0.05
		2.00	1.00	1.5	0.50	0.25
		4.00	1.00	0.9	-0.10	0.01
		2.00	1.00	2.25	1.25	1.56
		4.00	1.00	2.025	1.03	1.05

objective	$\sum ((y-y')/y')^2$	3.26

geometric mean error	$(\Pi(y/y'))^{1/n}$	4.90
average % error	$\Sigma(1/n)(100 * (y-y')/y')$ %	34%

Exercise Procedures

- **Run the Solver to find *a*,b** that minimize the objective**



Visit http://www.solver.com/tutorial.htm for background info

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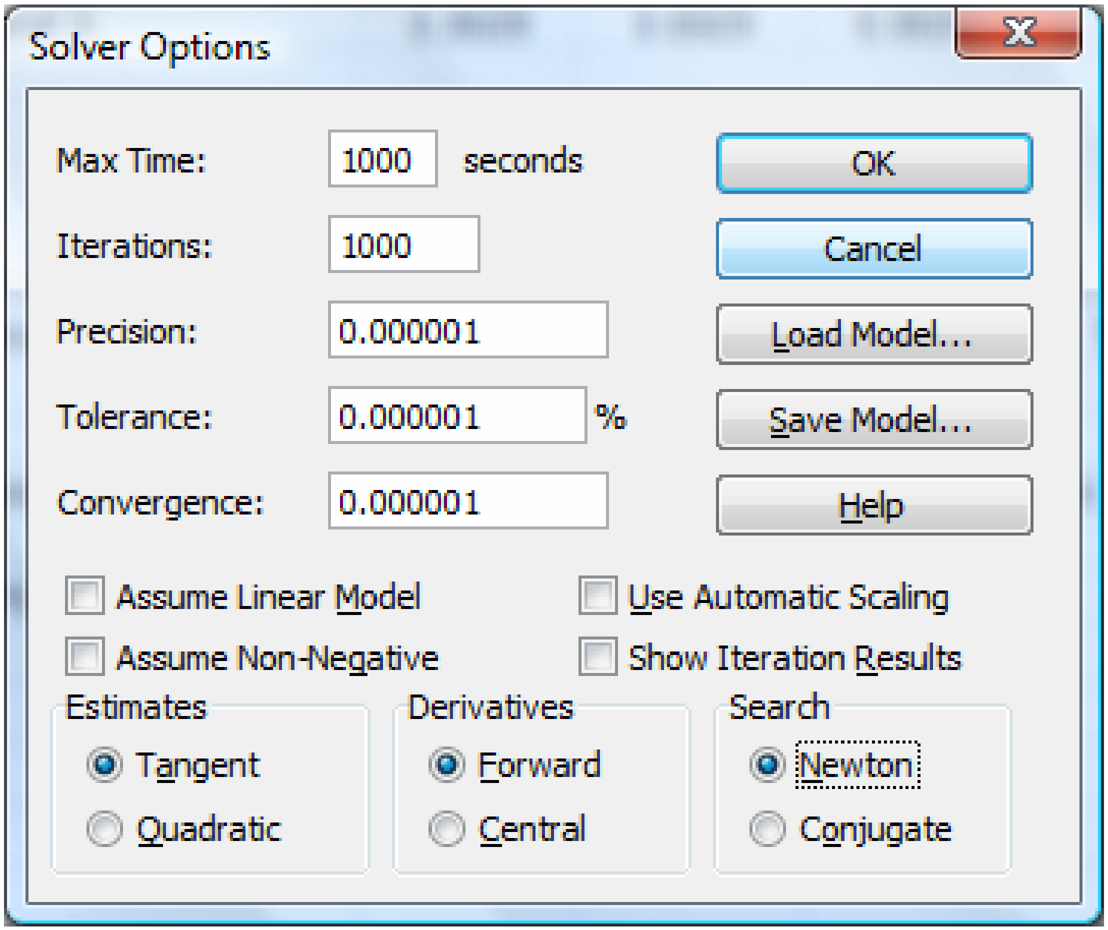
Exercise Procedures

- **Set the Solver options**



Visit http://www.solver.com/tutorial.htm for background info

Exercise Procedures

- **Record the results for different initial values for *a'*,*b'***

Summary area

Initial trial values	start *a'*	2	1	10	0.10
	start *b'*	0.75	1.00	10.00	0.10
Solver-found best values	optimal *a**	1.93	1.93	1.93	1.93
	optimal *b**	0.86	0.86	0.86	0.86
Initial & final objective values	start objective	11.01	26.11	62.97	5.41E+07
	optimal objective	8.80	8.80	8.80	8.80

Exercise Summary Template

Model function

Data set x values equally represented in all (a,b) pairs

Data set (a,b) pairs

Example: Data set includes $y = 2.5 \times 0.9^{x}$

$y=ab^{x}$, $\Sigma((y-y')/y')^{2}$, $x \in \{1,2\}$

(a', b')	(1.5, 0.6)	(1.5, 0.9)	(2.5, 0.6)	(2.5, 0.9)
lines	16	16	16	16

Objective function: Find (optimal) a^*,b^* minimizing this

Number of occurrences in the data set

… repeated 8 times with x = 1 and 8 with x = 2

- ***a'*,*b'*** are trial values for ***a*,*b***; ***y'*** is the associated y value
- ***a**,*b**** are trial ***a*,b** values that minimize the objective function; ***y**** is the associated "best" ***y* value**

Results Overview

Results Overview (1)

- *a*,b** were always different from the average a,b in the data set
  - *b* > Б* when we used the objective function $\Sigma((y-y')/y')^2$
  - *b* < Б* when we used the objective function $\Sigma((y-y')/y)^2$
- On the average, y*>y when we used the objective function $\Sigma((y-y')/y')^2$ and y*<y with the objective function $\Sigma((y-y')/y)^2$
- The *a',b'* we used to start the Solver tended to not make a difference in the Solver-identified *a*,b**, but
  - In some cases the Solver identified a "false *a*,b**"; but restarting it from this "false" *a*,b** led to an improved *a*,b**
  - Reducing the "convergence parameter" in the Solver options avoided this "early convergence"

Results Overview (2)

- Objective functions incorporating absolute values of *y-y'* were often ill-behaved in some areas
  - Initial *a',b'* made a difference in the Solver-identified *a*,b** in every case

How Do a,b* Relate to ā,Б?*

a^*, b^* versus ā, Б : Design for $a \times b^x$ & $\Sigma((y-y')/y')^2$

Design

$y=ab^x$, $\Sigma((y-y')/y')^2$, $x \in \{1,2\}$

(a′, b′)	**(1.5, 0.6)**	**(1.5, 0.9)**	**(2.5, 0.6)**	**(2.5, 0.9)**
lines	16	16	16	16

ā = 2.0, Б = 0.75

$x = \log_2 q$; $q \in \{2, 4\}$

- $a\,b^x$ generates the y value and $a'\,x^{b'}$ generates the y' values
- The best values of a and b minimize the sum of the squared "percentage errors," where the "percentage error" is relative to the estimated *y'* – versus the actual *y*
- Among the 64 data lines
  - 8 are $y = 1.5 \times 0.6^1$
  - 8 are $y = 1.5 \times 0.6^2$
  - 8 are $y = 1.5 \times 0.9^1$
  - 8 are $y = 1.5 \times 0.9^2$
  - etc.

Noteworthy: We've subsequently seen that one replicate per *a',b'* pair gives the same results as eight

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*a**,*b** and *ā*,*Б*: More About $a \times b^x$ & $\Sigma((y-y')/y')^2$

Summary area

Initial trial values	start *a'*	2	1	10	0.10
	start *b'*	0.75	1.00	10.00	0.10
Solver-found best values	optimal *a**	1.93	1.93	1.93	1.93
	optimal *b**	0.86	0.86	0.86	0.86
Initial & final objective values	start objective	11.01	26.11	62.97	5.41E+07
	optimal objective	8.80	8.80	8.80	8.80

- We provide the Solver four different sets of initial trial values, *a'* and *b'* – in case this affects the *a** and *b** values it finds

  The first initial trial values are the average *a* and the average *b*

- For each *a'* and *b'* trial value in Solver there is a value of the objective function, $\Sigma((y-y')/y')^2$ where $y' = a' \times b'^{\,x}$ for the *x* value on each data line

  The start objective is the value of the objective at the initial a',b'

a,b* and ā, Ƃ : Results for a×b^x & Σ((y-y')/y')^2*

	Summary area				
Initial trial values	start *a'*	2	1	10	0.10
	start *b'*	0.75	1.00	10.00	0.10
Solver-found best values	optimal *a**	1.93	1.93	1.93	1.93
	optimal *b**	0.86	0.86	0.86	0.86
Initial & final objective values	start objective	11.01	26.11	62.97	5.41E+07
	optimal objective	8.80	8.80	8.80	8.80

We see that

- Initial trial values don't affect the Solver-identified *a** , *b**
- *a*, b** improve the objective compared to all initial *a'*, *b'*, including the average *a, b*
- Solver-identified *a*, b** are respectively less and greater than the average *a* and *b*

Background

$$a \times b^{\log_2 x} = a \times x^{\log_2 b}$$

a*,b* and ā,Б: Design for $a \times x^b$ & $\Sigma((y-y')/y')^2$

Design

$y=ax^b$, $\Sigma((y-y')/y')^2$, $x \in \{2,4\}$

(a', b')	(1.5, -0.15)	(1.5, -0.74)	(2.5, -0.15)	(2.5, -0.74)
lines	16	16	16	16

$\bar{a} = 2.0$, Б = -0.44

$b = \log_2 r$; $r \in \{0.60, 0.90\}$

- $a\,x^b$ generates the y values – prior design used $a\,b^x$
- The best values of *a* and *b* minimize the sum of the squared "percentage errors," where the "percentage error" is relative to the "estimated" *y'* – versus the "actual" *y*
- Among the 64 data lines
  - 8 are $y = 1.5 \times 0.2^{-0.15}$
  - 8 are $y = 1.5 \times 0.4^{-0.15}$
  - 8 are $y = 1.5 \times 0.2^{-0.74}$
  - 8 are $y = 1.5 \times 0.4^{-0.74}$
  - etc.,

Noteworthy: We've subsequently seen that one replicate per *a',b'* pair gives the same results as eight

a*,b* and ā,Б : More About $a \times x^b$ & $\Sigma((y-y')/y')^2$

Design

$y=ax^b$, $\Sigma((y-y')/y')^2$, $x \in \{2,4\}$

(a', b')	(1.5, -0.15)	(1.5, -0.74)	(2.5, -0.15)	(2.5, -0.74)
lines	16	16	16	16

ā = 2.0, Б = -0.44

$b = \log_2 r$; $r \in \{0.60, 0.90\}$

- The *x, b* values here are related to the *x, b* values in the $y = a\, b^x$ work
  - *b* values **here** are $\log_2$ of those before: $\log_2 0.6 \approx -0.74$, $\log_2 0.9 \approx -0.15$; or $2^{-0.74} \approx 0.6$, $2^{-0.15} \approx 0.9$
  - *x* values **before** are $\log_2$ of *x* values here: $\log_2 2 = 1$, $\log_2 4 = 2$ – or $2^1 = 2$, $2^2 = 4$
- The *a* values are the same

When we relate the *b*s and *x*s this way – exponents in one form are $\log_2$ of the base values in the other form – we can show that the two formulas are equivalent and y values are identical – i.e., equivalent ways of stating a learning curve. Allows us to ask if we get the same *a*,b** despite “surface differences” in the equation

Design

$y=ab^x$, $\Sigma((y-y')/y')^2$, $x \in \{1,2\}$

(a', b')	(1.5, 0.6)	(1.5, 0.9)	(2.5, 0.6)	(2.5, 0.9)
lines	16	16	16	16

ā = 2.0, Б = 0.75

$x = \log_2 q$; $q \in \{2, 4\}$

a*,b* and ā,Б : Results for $a \times x^{b}$ & $\sum((y-y')/y')^2$

	Summary area				
Initial trial values	start a *	2	1	10	0
	start b *	-0.42	0	3	2
Solver-found best values	optimal a *	1.93	1.93	1.93	1.93
	optimal b *	-0.22	-0.22	-0.22	-0.22
Initial & final objective values	start objective	11.01	26.11	62.97	12.43
	optimal objective	8.80	8.80	8.80	8.80

We see that

- Initial trial values don't affect the Solver-identified *a** and *b**
- *a*,b** improve the objective compared to that for the average *a* and *b*
- Solver-identified a*,b* are respectively less and greater than the average a and b
- *a*,b** are identical to $y=a \times b^{X}$ solutions after undoing $\log_2$ transforms – $2^{-0.22} \approx 0.86$
- Not immediately obvious the Solver would find the related *b** values across the equivalent but different equation forms for *y*

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Earlier Results for a $\times$b ^{x} & $\Sigma((y-y')/y')^2$

	Summary area				
Initial trial values	start *a'*	2	1	10	0.10
	start *b'*	0.75	1.00	10.00	0.10
Solver-found best values	optimal *a**	1.93	1.93	1.93	1.93
	optimal *b**	0.86	0.86	0.86	0.86
Initial & final objective values	start objective	11.01	26.11	62.97	5.41E+07
	optimal objective	8.80	8.80	8.80	8.80

a*,b* and ā,Б: Set All a=2 Exercise

Set all *a* = 2 in the data set – which should further simplify the Solver's problem – with analogous results

- Initial trial values don't affect the Solver-identified *a** and *b**
- *a**,*b** improve the objective compared to that for the average a and b
- *a**,*b** are identical in $y = a \times x^b$ and $y = a \times b^x$ variants after undoing the $\log_2 b$ transform
- Solver-identified *b** is greater than the average *b*; *a** is less than 2, the average *a*
  - *a** = 1.81 is slightly less than before; *b** = 0.86 or -0.22 is identical

Design

$y=ab^x$, $\Sigma((y-y')/y')^2$, $x \in \{1,2\}$

(a', b')	**(2, 0.6)**	**(2, 0.9)**	**(2, 0.6)**	**(2, 0.9)**
lines	16	16	16	16

$\bar{a}$ = 2.0, Б = 0.75

$x = \log_2 q$; $q \in \{2, 4\}$

Design

$y=ax^b$, $\Sigma((y-y')/y')^2$, $x \in \{2,4\}$

(a', b')	**(2, -0.15)**	**(2, -0.74)**	**(2, -0.15)**	**(2, -0.74)**
lines	16	16	16	16

$\bar{a}$ = 2.0, Б = -0.44

$b = \log_2 r$; $r \in \{0.60, 0.90\}$

a,b* and ā,Б: Results for Σ((y-y')/y')*^{8}

Analogous results in designs using a sum of eighth power errors

- Initial trial values don't affect the Solver-identified *a** and *b**
- *a*,b** improve the objective compared to that for the average a and b
- *a*,b** are identical in $y = a \times x^b$ and $y = a \times b^x$ variants after undoing the $\log_2 b$ transform
- Solver-identified *b** is greater than the average *b*; *a** is less than the average *a*

Design

$y = a\,b^x$, $\Sigma((y-y')/y')^8$, $x \in \{1,2\}$

(*a'*, *b'*)	(1.5, 0.6)	(1.5, 0.9)	(2.5, 0.6)	(2.5, 0.9)
lines	16	16	16	16

$\bar{a}$ = 2.0, Б = 0.75

$x = \log_2 q$; $q \in \{2, 4\}$

Design

$y = ax^b$, $\Sigma((y-y')/y')^8$, $x \in \{2,4\}$

(*a'*, *b'*)	(1.5, -0.15)	(1.5, -0.74)	(2.5, -0.15)	(2.5, -0.74)
lines	16	16	16	16

$\bar{a}$ = 2.0, Б = -0.44

$b = \log_2 r$; $r \in \{0.60, 0.90\}$

Design

$y = a\,b^x$, $\Sigma((y-y')/y')^8$, $x \in \{1,2\}$

(*a'*, *b'*)	(2, 0.6)	(2, 0.9)	(2, 0.6)	(2, 0.9)
lines	16	16	16	16

$\bar{a}$ = 2.0, Б = 0.75

$x = \log_2 q$; $q \in \{2, 4\}$

Design

$y = a\,x^b$, $\Sigma((y-y')/y')^8$, $x \in \{2,4\}$

(*a'*, *b'*)	(2, -0.15)	(2, -0.74)	(2, -0.15)	(2, -0.74)
lines	16	16	16	16

$\bar{a}$ = 2.0, Б = -0.44

$b = \log_2 r$; $r \in \{0.60, 0.90\}$

a*,b* and ā,Б: Results for "unbalanced bs"

Design $y=ab^x$, $\Sigma((y-y')/y')^2$, $x\in\{1,2\}$

(a′, b′)	(1.5, 0.6)	(1.5, 0.9)	(2.5, 0.6)	(2.5, 0.9)
lines	24	8	24	8

ā = 2.0, Б = **0.675**

$x = \log_2 q;\ q \in \{2, 4\}$

Design $y=ab^x$, $\Sigma((y-y')/y')^2$, $x\in\{1,2\}$

(a′, b′)	(1.5, 0.6)	(1.5, 0.9)	(2.5, 0.6)	(2.5, 0.9)
lines	8	24	8	24

ā = 2.0, Б = **0.825**

$x = \log_2 q;\ q \in \{2, 4\}$

Design $y=ax^b$, $\Sigma((y-y')/y')^2$, $x\in\{2,4\}$

(a′, b′)	(1.5, -0.74)	(1.5, -0.15)	(2.5, -0.74)	(2.5, -0.15)
lines	24	8	24	8

ā = 2.0, Б = **-0.591**

$b = \log_2 r;\ r \in \{0.60, 0.90\}$

Design $y=ax^b$, $\Sigma((y-y')/y')^2$, $x\in\{2,4\}$

(a′, b′)	(1.5, -0.74)	(1.5, -0.15)	(2.5, -0.74)	(2.5, -0.15)
lines	8	24	8	24

ā = 2.0, Б = **-0.278**

$b = \log_2 r;\ r \in \{0.60, 0.90\}$

Analogous results in sum of "unbalanced-*b*s," squared error designs

- Initial trial values don't affect the Solver-identified *a** and *b**
- *a**,*b** improve the objective compared to that for the average *a* and *b*
- Solver-identified a*, b* are both greater than the average *a* and *b*
- *a**,*b** are identical in $y = a \times x^b$ and $y = a \times b^x$ variants after undoing the $\log_2 b$ transform

How Do a*,b* Relate to ā,Б? Summary

Function	Objective function	Other	$\bar{a}$	a^*	Б:	b^*	Notes
$y = a\,b^{x}$	$\Sigma\,((y-y')/y')^2$	$a \in \{1.5, 2.5\}$ $b \in \{0.6, 0.9\}$	2	1.925	0.75	0.861	
		all a=2	2	1.812	0.75	0.861	
	$\Sigma\,((y-y')/y')^8$	$a \in \{1.5, 2.5\}$ $b \in \{0.6, 0.9\}$	2	1.940	0.750	0.834	
		all a=2	2	1.903	0.750	0.793	analogous results for $y = a\,x^{b}$
	$\Sigma\,((y-y')/y')^2$	$a \in \{1.5, 2.5\}$ $b \in$ {0.6, 0.9: 3:1}	2	1.883	0.675	0.790	
	$\Sigma\,((y-y')/y')^2$	$a \in \{1.5, 2.5\}$ $b \in$ {0.6, 0.9: 1:3}	2	2.020	0.825	0.889	

Noteworthy

- *a** and *b** stay within the range of given values
- *b** stays within [*Б*, max(*b*)]; *a** generally stays within [min(*a*), *ā*]

On the average ⊕	When the objective is:
$y^* > y$	$\Sigma((y-y')/y')^2$
$y^* < y$	$\Sigma((y-y')/y)^2$

⊕ **Arithmetic mean** $\Sigma\,(1/n)\,(y - y^*) < 0$

Geometric mean $\Pi\,(y/y^*)^{1/n} < 1$

$y^* > y$ When $\Sigma((y-y')/y')^2$: Examples

Design

$y=ab^x$, $\Sigma((y-y')/y')^2$, $x \in \{1,2\}$

(a′, b′)	(1.5, 0.6)	(1.5, 0.9)	(2.5, 0.6)	(2.5, 0.9)
lines	16	16	16	16

$\bar{a} = 2.0$, Б = 0.75

$x = \log_2 q$; $q \in \{2, 4\}$

Design

$y=ax^b$, $\Sigma((y-y')/y')^2$, $x \in \{2,4\}$

(a′, b′)	(1.5, -0.15)	(1.5, -0.74)	(2.5, -0.15)	(2.5, -0.74)
lines	16	16	16	16

$\bar{a} = 2.0$, Б = -0.44

$b = \log_2 r$; $r \in \{0.60, 0.90\}$

Summary area for both cases

start a	2.00	1.00	10.00	0.10
start b	0.75	1.00	10.00	0.10
optimal a	1.925	1.925	1.925	1.925
optimal b	0.861	0.861	0.861	0.861
$\Pi(y/y^*)^{1/n}$	0.79	0.79	0.79	0.79
$\Sigma(1/n)(y-y^*)$	-0.21	-0.21	-0.21	-0.21

Noteworthy

- Also *b* > Б*

But $y^* < y$ When $\Sigma((y-y')/y)^2$: Examples

Design

$y = a\, b^x$, $\Sigma((y-y')/y)^2$, $x \in \{1,2\}$

(a′, b′)	(1.5, 0.6)	(1.5, 0.9)	(2.5, 0.6)	(2.5, 0.9)
lines	16	16	16	16

$\bar{a}$ = 2.0, Б = 0.75

$x = \log_2 q$; $q \in \{2, 4\}$

Design

$y = a\, x^b$, $\Sigma((y-y')/y)^2$, $x \in \{1,2\}$

(a′, b′)	(1.5, -0.15)	(1.5, -0.74)	(2.5, -0.15)	(2.5, -0.74)
lines	16	16	16	16

$\bar{a}$ = 2.0, Б = -0.44

$b = \log_2 r$; $r \in \{0.60, 0.90\}$

Summary area

start a	2.00	1.00	10.00	0.10
start b	0.75	1.00	10.00	0.10
optimal a	1.948	1.948	1.948	1.948
optimal b	0.627	0.627	0.627	0.627
$\Pi(y/y^*)^{1/n}$	1.26	1.26	1.26	1.26
$\Sigma(1/n)(y-y^*)$	0.34	0.34	0.34	0.34

Noteworthy

- Also ***b* < Б***

But $y^* < y$ When $\Sigma((y-y')/y)^2$: More Examples

Design

$y=a\,b^x,\ \Sigma((y-y')/y)^8,\ x \in \{1,2\}$

(a′, b′)	(1.5, 0.6)	(1.5, 0.9)	(2.5, 0.6)	(2.5, 0.9)
lines	16	16	16	16

ā = 2.0, Б = 0.75

$x = \log_2 q;\ q \in$ {2,

Design

$y=a\,b^x,\ \Sigma((y-y')/y)^2,\ x \in \{1,2\}$

(a′, b′)	(1.5, 0.6)	(1.5, 0.9)	(2.5, 0.6)	(2.5, 0.9)
lines	24	8	24	8

ā = 2.0, Б = **0.675**

$x = \log_2 q;\ q \in$ {2,

Design

$y=a\,b^x,\ \Sigma((y-y')/y)^2,\ x \in \{1,2\}$

(a′, b′)	(1.5, 0.6)	(1.5, 0.9)	(2.5, 0.6)	(2.5, 0.9)
lines	8	24	8	24

ā = 2.0, Б = **0.825**

$x = \log_2 q;\ q \in$ {2,

|(y-y’)/•|: Absolute Value Objectives ⊕

⊕ “•” is y or y’

|(y-y')/• |: Absolute Value Objectives ⊕ (1)

- Absolute value objective functions give equal weight to *y-y'* differences in determining *a**,*b**
- |(*y-y**) /• | and its equivalent, ((*y-y**) /•)^2)$^{0.5}$, gave identical results
- Some results followed the patterns we observed earlier
  - *ā,Б* were never the best values for *a,b*
    - Solver solutions *a*,b** always had smaller objective function values
  - |(*y-y'*) /*y'* | led to *y** > *y* on the average

    |(*y-y'*) /***y*** | led to *y****<*** *y* on the average

⊕ ***"•" is y or y'***

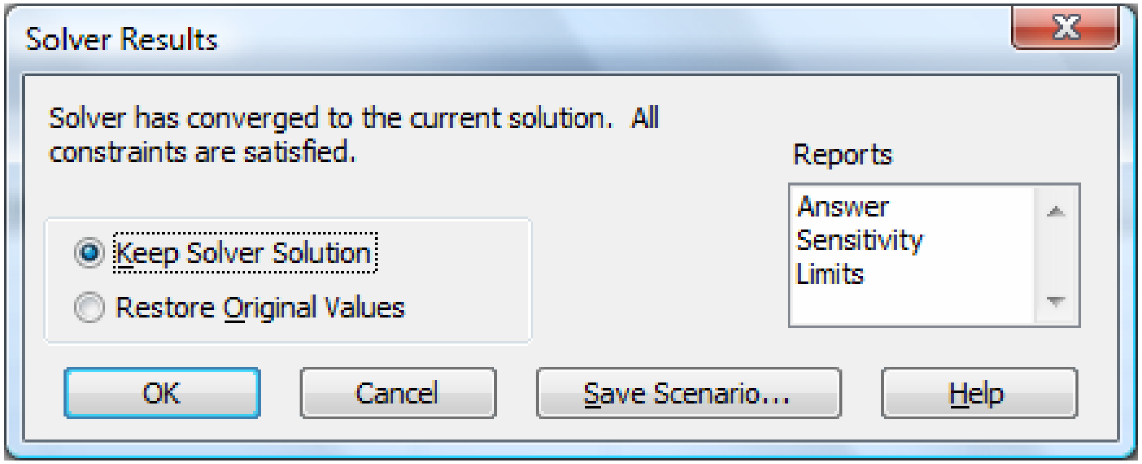
|(y-y')/• |: Absolute Value Objectives (2)

- Other results didn't follow earlier patterns
  - Initial values for *a',b'* mattered for both y* and y as "•"
    - Each initial *a',b'* led to different *a*,b**
    - *ā,Б* initial values gave *a*,b** with the best objective function values
  - Solver found different *a*,b** for $y = a \times b^x$ and for $y = a \times x^b$
- These behaviors may pose challenges to those interested in absolute value-type objective functions

Solver Stops Early on Nonoptimal a,b**

Solver Stops Early on Nonoptimal a,b**

- Is the Solver-identified *a*,b** the best solution?



- Not always. We saw cases of "early convergence" to a "false" *a*,b**
  - The Solver found a single *a*,b** for three of the initial *a',b'*; with a better objective than the "false" *a*,b**
  - Running the Solver again with the "false" *a*,b** as the initial *a',b'* found the common *a*,b**
  - We observed early convergence under $\Sigma((y\text{-}y')/y')^2$ for $y = a \times b^x$ with initial *a',b'* = 0.1,0.1; and for $y = a \times x^b$ with *a',b'* = 0.1, $\log_2 0.10 \approx -3.32$)
  - Under $\Sigma((y\text{-}y')/y)^2$ it occurred with10,10 and with 10, $\log_2 10 \approx 3.32$, respectively

Solver Stops Early: Example

Design

$y = a\, b^{x}$, $\Sigma((y-y')/y)^2$, $x \in \{1,2\}$

(a', b')	**(1.5, 0.6)**	**(1.5, 0.9)**	**(2.5, 0.6)**	**(2.5, 0.9)**
lines	16	16	16	16

$\bar{a} = 2.0,\ \bar{b} = 0.75$

$x = \log_2 q;\ q \in \{2, 4\}$

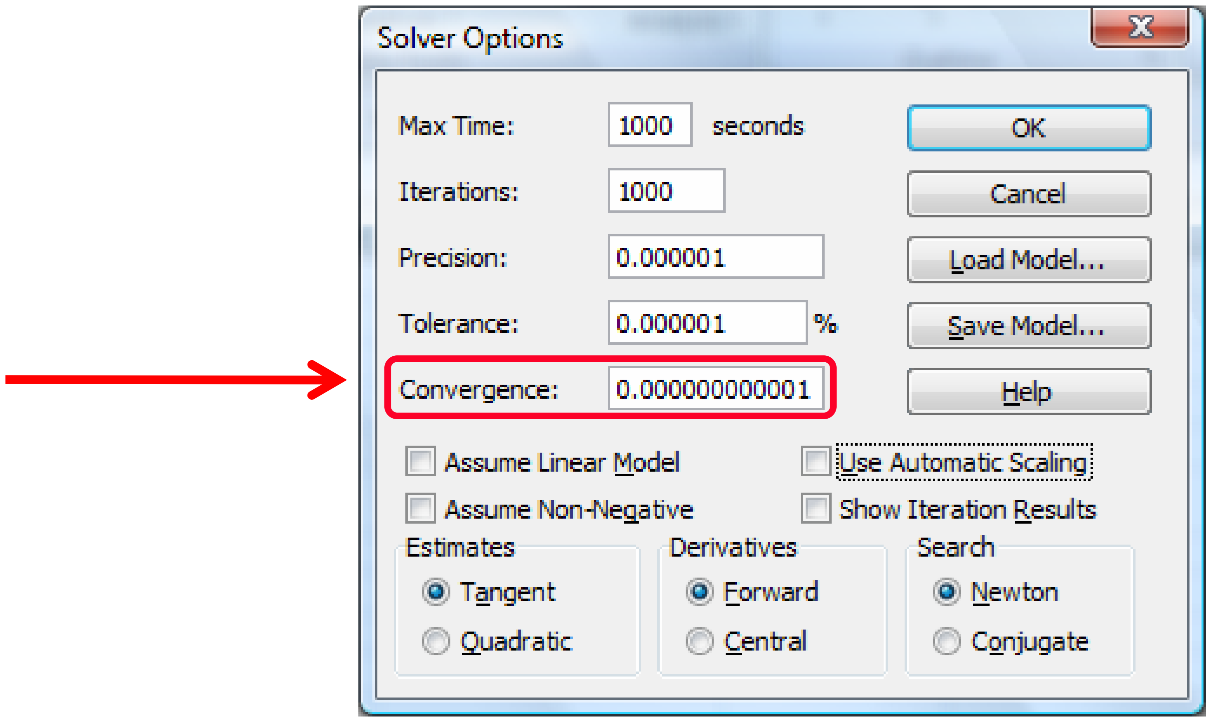
- From *a'*,*b'* = 0.10, 0.10, the Solver converges to false a*,b* = 0.954, 1.494
- From *a'*,*b'* = 0.954, 1.494, the Solver converges to the common *a**,*b** = 1.925, 0.861; the objective function improves from 12.43 to the common 8.80

Summary area

start a	2.00	1.00	10.00	0.10	0.954
start b	0.75	1.00	10.00	0.10	1.494
optimal a	1.925	1.925	1.925	0.954	1.925
optimal b	0.861	0.861	0.861	1.494	0.861
optimal objective	8.80	8.80	8.80	12.43	8.80

When the Solver Stops Early: One Clue (1)

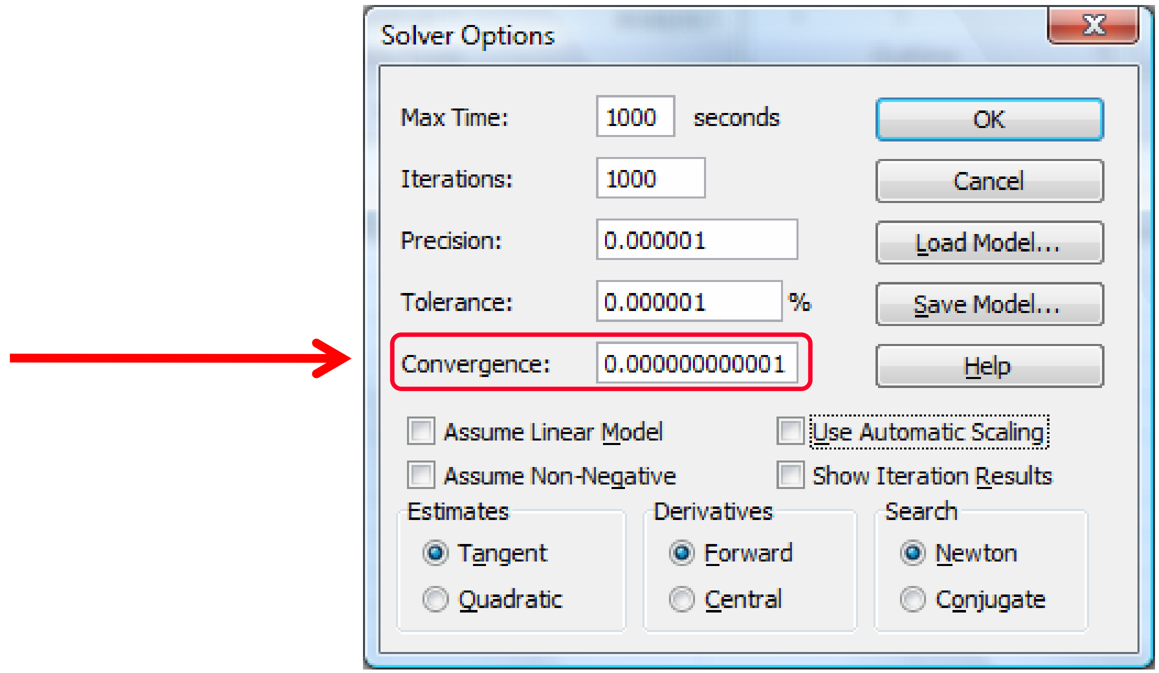
- Reducing the convergence constant from 10^{-6} to 10^{-12} avoided the false *a*,b**



- The convergence constant provides a stopping threshold; the Solver "converges" if the relative change in the solution is no greater than the threshold (http://office.microsoft.com/en-us/excel/HP100726911033.aspx)

When the Solver Stops Early: One Clue (2)

- We conjectured that early convergence reflected a relative "flat zone" in the objective function surface – versus a local minimum



- Reducing the convergence constant allows the Solver to continue solving in the presence of only "modest" improvements in the objective

When the Solver Stops Early: One Clue (3)

- We plotted the *a',b'* over Solver iterations

Design

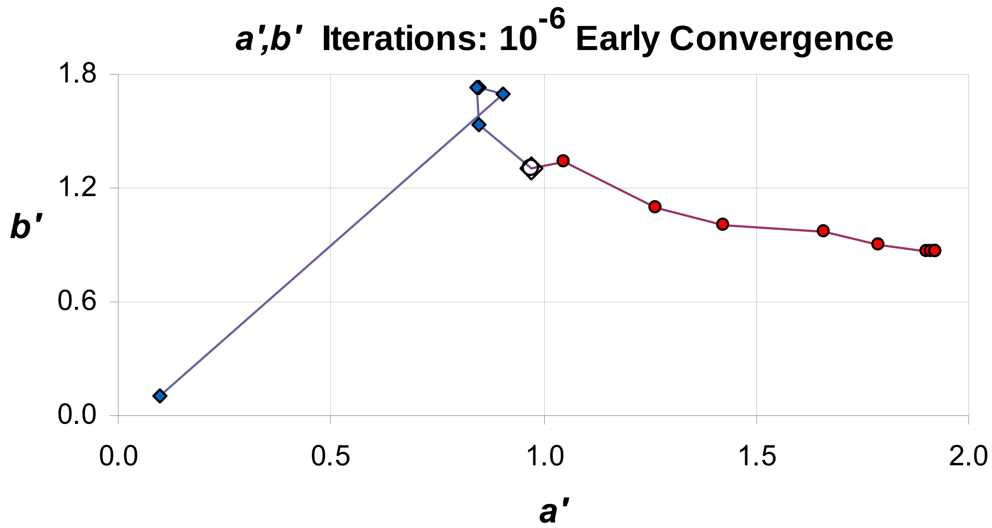
$y=ab^x$, $\Sigma((y-y')/y')^2$, $x \in \{1,2\}$

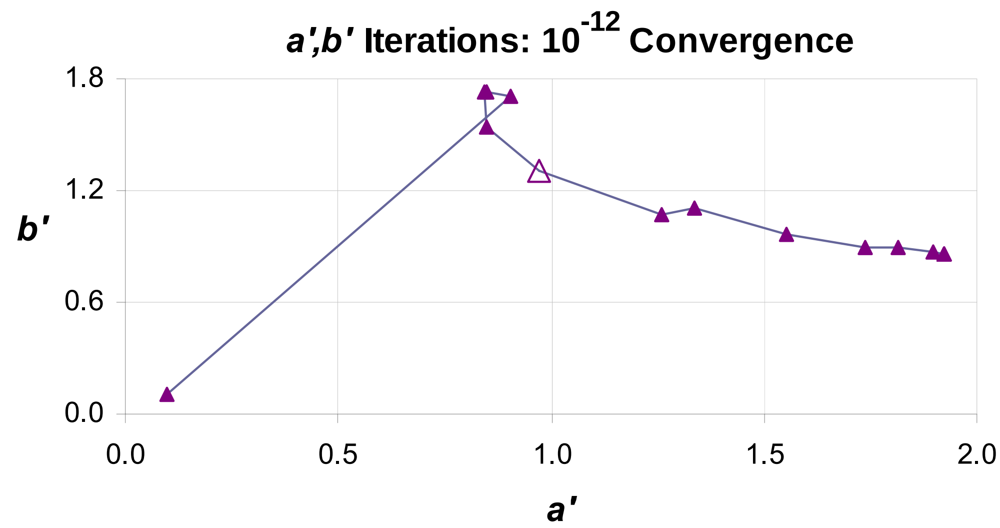
(a', b')	**(1.5, 0.6)**	**(1.5, 0.9)**	**(2.5, 0.6)**	**(2.5, 0.9)**
lines	16	16	16	16

$\bar{a} = 2.0$, $\bar{b} = 0.75$

$x = \log_2 q$; $q \in \{2, 4\}$

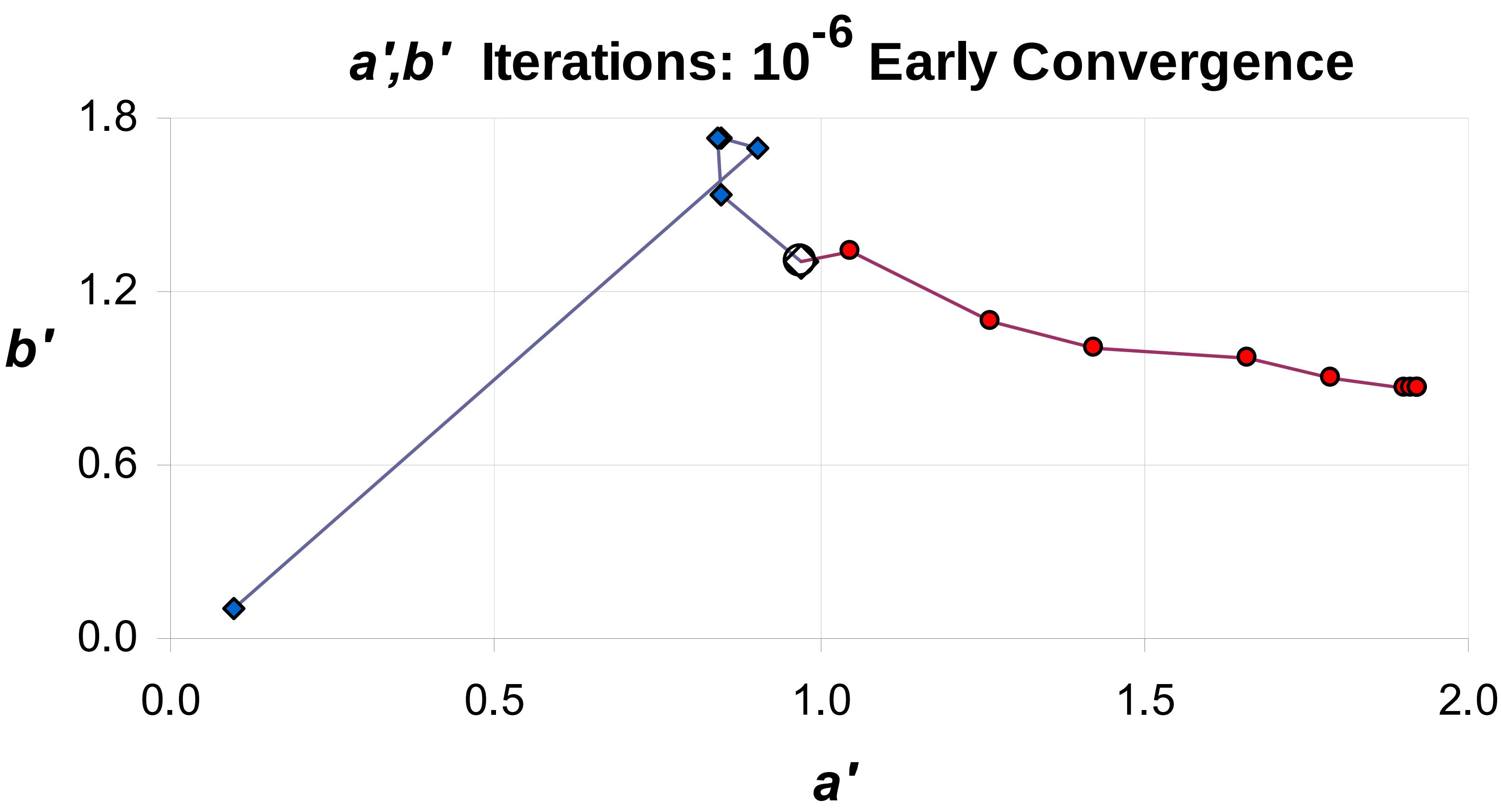
When the Solver Stops Early: One Clue (3)



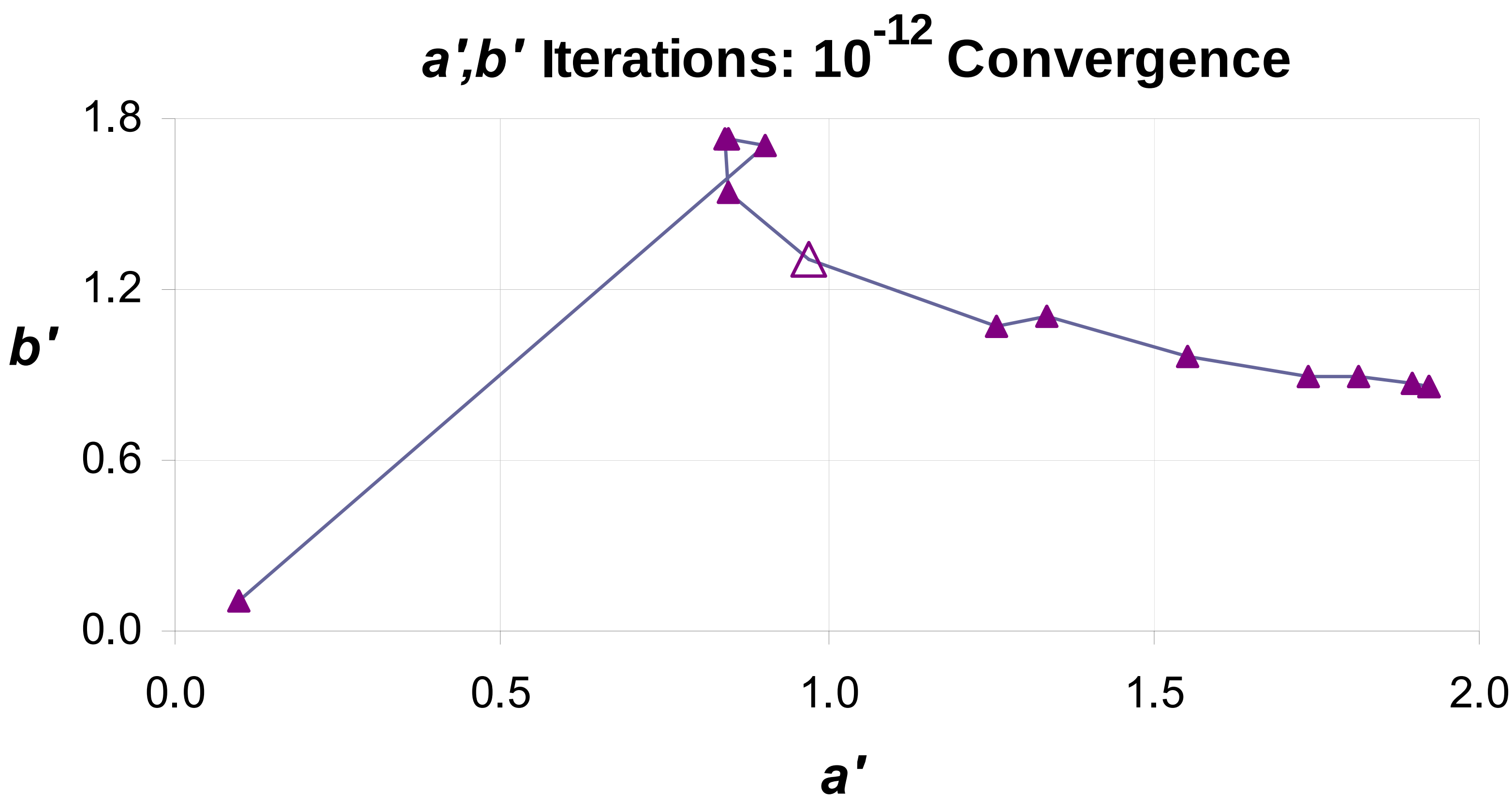


- Solving with 10^{-12} convergence finds the common *a*,b** in one run
- The two *a',b'* paths are similar but not identical after the restart point – compare the eighth through fifteenth points on both paths
- 10^{-12} convergence stops in one fewer iteration

When the Solver Stops Early: One Clue (4)



When the Solver Stops Early: One Clue (5)



Two Cases of Lognormal Data

What About Lognormal Data ?(1)

- **Previously worked with uniform *x*s in the data**
  - ***x* = 2 or 4 in equal numbers, *b* = 0.6 or 0.9 in varying mixtures**

Design

$y=ab^{x}$, $\Sigma((y-y')/y')^{2}$, $x \in \{1,2\}$

(a′, b′)	(1.5, 0.6)	(1.5, 0.9)	(2.5, 0.6)	(2.5, 0.9)
lines	16	16	16	16

ā = 2.0, Б = 0.75

$x = \log_2 q$; $q \in \{2, 4\}$

  - **Results roughly met our "naive" expectations**
    - ***b**** **greater than *Б* but tracked changes to *Б* as we varied the mixture of 0.6s and 0.9s in the data set**

What About Lognormal Data ?(2)

- In this excursion we work with *x*s that are a lognormal sample or are derived from lognormal samples

Design

$y=ab^{x}$, $\Sigma((y-y')/y')^2$

(a', b')	(1.5, 0.6)	(1.5, 0.9)	(2.5, 0.6)	(2.5, 0.9)
lines	20 (*x*s)	21 (*x*s)	22 (*x*s)	23 (*x*s)

$\bar{a}$ = 2.0, $\bar{b}$ = 0.75

$x=\log_2(q \text{ samples})$

$q \sim ln(\mu=2, \sigma = 1)$

Design

$y=ax^{b}$, $\Sigma((y-y')/y')^2$

$x\sim \ln(\mu=2, \sigma=1)$

(a', b')	(1.5, -0.15)	(1.5, -0.74)	(2.5, -0.15)	(2.5, -0.74)
lines	20 (*x*s)	21 (*x*s)	22 (*x*s)	23 (*x*s)

$\bar{a}$ = 2.0, $\bar{b}$ = -0.44

$b = \log_2 r$; $r \in \{0.60, 0.90\}$

What About Lognormal Data ?(3)

- Results were generally consistent with earlier results
  - The common b^* for $a \times b^x$ was greater than the average b
  - The common a^*, b^* was identical for $a \times b^x$ and $a \times x^b$ after undoing the $\log_2 b$ transform
- Unlike the earlier results
  - The common a^* was greater than the average a

Summary *(1)*

- We've observed a number of behaviors in a simple environment - which may or may not generalize to "practical" environments or to more general environments. Among them:
  - When we used a $\Sigma(y - y')/y')^2$ objective function, *b** was greater than the average *b*; and *y** > *y*

    We reversed these results for for a $\Sigma((y - y^*)/y)^2$ objective
  - The average a and b were always good starting points
  - The Solver converged early for some initial a',b'; restarting the Solver from the "false" *a*,b** found the common – and better – *a*,b**

  We avoided the early stop by reducing the Solver's "convergence constant"

  We made it practice to re-run the Solver several times in all cases; one rerun was usually enough – but not always

Summary (2)

- We've observed a number of behaviors in a simple environment - which may or may not generalize to "practical" environments or to more general environments. Among them:
  - Initial *a',b'* did not affect the final results with $\Sigma((y\text{-}y^*)/y^*)^k$, $k \in (2,8)$
  - Initial *a',b'* did affect results when we used an absolute value objective function – $\Sigma|(y\text{-}y^*)/y^*|$ or $\Sigma(((y\text{-}y^*)/y^*)^2)^{0.5}$ – Solver identified a different *a*,b** for each initial *a',b'*

Moving Forward

- Catalog key theoretical concepts on fitting power and exponential functions
    - Use these theory to drive further empirical tests
- Catalog lessons learned from these exercises
    - Use these empirical results to drive investigations into theory
- Address the question of when and why you should want to use power and exponential functions
    - We should want *a priori* conditions
    - Statistical goodness-of-fit tests provide *a posteriori* rationale; may be “hijacked” by “unusual data”; and still leave open the “why” question
    - The math/economics/psychology literature on measurement scale theory provides a solid starting point

The End

Additional Discussions

Why We Studied "Nonlinear Methods"
and the Solver in Particular

- **Why study iterative, "only approximate," clearly imperfect methods for fitting model constants when we can always solve linear, log-transformed models?**
  - **Not all interesting CERs with exponential and power form components will be "log transformable"**
  - **Log transform approaches have their own properties - which are not in the scope of this report - not all of which may be desirable**
  - **Iterative methods, including the Solver, have a user community convinced about their utility**
  - **The project goal is to document methods' characteristics and not to make the case for one method or another as being the best**

A Subtlety

- **In "ordinary" regression analyses we:**
  - **assume a single value for *a* and for *b* drive the *y*s**
  - **disturbances in the *y*s we observe prevent us from identifying *a* and *b* exactly; we can only estimate *a* and *b* only with uncertainty**
  - **we term the data "noisy but homogeneous"**
- **Here, the regression data are perfect but heterogeneous:**
  - **each line has varying *a*s and *b*s**
  - **we exactly calculate *y* on each line from the given *a, b,* and *x* values**
  - **we search for a single value for *a* and for *b* that together, "best represent" the differing *a*,*b* pairs in the data set**
  - **the errors are due, in principle, to the inability of any one a, b pair to represent the differing *a*,*b* pairs in the data**
  - **when the data set comprises different types of articles we consider the data set heterogeneous**